LOCALLY FLAT, LOCALLY TAME, AND TAME EMBEDDINGS

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1. **Introduction.** Brown [1] has shown that an S^{n-1} embedded in a locally flat manner in S^n is flat and hence tame in S^n. Bing [2] and Moise [3] have shown that locally tame subsets of 3-manifolds are tame. However, in the general case, it is not known whether a manifold N embedded in a locally flat manner in a triangulated manifold M or a polyhedron P embedded in a locally tame manner in a triangulated manifold M are tame in M. Partial solutions to both of these problems have been obtained by the author and will be stated in §3 of this paper. I have been informed by R. H. Bing that Herman Gluck has obtained similar results.

2. **Definitions and notations.** Let N^k be a combinatorial k-manifold. Then $(N^k)^r$ will denote the rth barycentric subdivision of N^k. If α is a k-simplex of $(N^k)^r$ and α'' is the union of all simplexes of $(N^k)^{r+2}$ contained in α, then C_α will denote the closed simplicial neighborhood of $|\alpha''|$, the polyhedron of α'', in $(N^k)^{r+2}$. That is C_α is the union of all closed simplexes in $(N^k)^{r+2}$ that meet $|\alpha''|$. Since α'' is collapsible, C_α is a combinatorial k-ball [4].

The statement that f is a locally flat embedding of a k-manifold N^k in an n-manifold N^n, means that each point of $f(N^k)$ has a neighborhood U in N^n such that the pair $(U, U\cap f(N^k))$ is homeomorphic to the pair (R^n, R^k).

Two definitions of locally tame will now be given.

DEFINITION 1. Let N be a manifold topologically embedded in a triangulated manifold M. N is locally tame if for each point p of N, there exists a neighborhood U of p in M and a homeomorphism h of $\overline{U}$ into M, such that $h[\mathrm{Cl}(U\cap N)]$ is a polyhedron in M.

DEFINITION 2. Let P be a polyhedron topologically embedded in a triangulated manifold M. P is locally tame if for each point p of P, there exists a neighborhood U of p in M and a homeomorphism h of $\overline{U}$ into M, such that $h|\mathrm{Cl}(U\cap P)$ is piecewise linear with respect to a fixed triangulation T of P.

Let K be a complex topologically embedded by f in a triangulated n-manifold N^n and let $\epsilon > 0$. Suppose there exists an ϵ-homeomorphism h of N^n onto itself such that if $U_\epsilon(f(K))$ denotes the set of points in N^n whose distance from $f(K)$ is less than ϵ, then

(i) $h|N^n - U_\epsilon(f(K)) = 1$,

(ii) $hf: K \to N^n$ is a piecewise linear embedding.

Then $f(K)$ will be said to be ϵ-tame in N^n.

3. Statement of results.

THEOREM 1. *Let f be a locally flat embedding of a closed combinatorial k-manifold N^k in a closed combinatorial n-manifold N^n, $2k+2 \leqq n$ and $\epsilon > 0$. Then $f(N^k)$ is ϵ-tame in N^n.*

THEOREM 2. *Let f_1 and f_2 be locally flat (locally tame) embeddings of a closed combinatorial k-manifold N^k (finite k-polyhedron P^k) in S^n and $2k+2 \leqq n$. Then there exists a homeomorphism h of S^n onto itself such that $hf_1 = f_2$.*

THEOREM 3. *Let f be a locally flat embedding of a k-manifold N^k in a combinatorial n-manifold N^n and $2k+2 \leqq n$. Then $f(N^k)$ is locally tame (Definition 1).*

THEOREM 4. *Let f be a locally tame (Definition 2) embedding of a possibly infinite k-polyhedron P^k as a closed subset of the interior of a combinatorial n-manifold N^n, $2k+2 \leqq n$ and $\epsilon > 0$. Then $f(P^k)$ is ϵ-tame in N^n.*

4. Reference theorems.

HOMMA'S THEOREM [5]. *Let M^n, $\hat{M}^n$ and $\hat{P}^k$ be two finite combinatorial n-manifolds and a finite polyhedron such that $\hat{M}^n$ is topologically embedded in M^n, $\hat{P}^k$ is piecewise linearly embedded in* $\text{Int}(\hat{M}^n)$ *and $2k+2 \leqq n$. Then for $\epsilon > 0$, $\hat{P}^k$ is ϵ-tame in M^n.*

GLUCK'S MODIFICATION OF HOMMA'S THEOREM [6]. *Let the following be given:*

(i) *M^n, a possibly noncompact combinatorial n-manifold;*

(ii) *$\hat{M}^n$, a possibly noncompact combinatorial n-manifold, topologically embedded in M^n;*

(iii) *$\hat{P}^k$, a possibly infinite polyhedron, piecewise linearly embedded as a closed subset of* $\text{Int}(\hat{M}^n)$;

(iv) *$\hat{L}$, a subpolyhedron of $\hat{P}^k$ such that* $\text{Cl}(\hat{P}^k - \hat{L})$ *is a finite polyhedron, and such that $\hat{L}$ is piecewise linearly embedded in M^n as well as in $\hat{M}^n$.*

If $2k+2 \leqq n$, then for any $\epsilon > 0$, there is an ϵ-homeomorphism F of M^n onto M^n such that under F, $\hat{P}^k - \hat{L}$ is ϵ-tame in M^n and $F|\hat{L} = 1$.

5. Partial proofs of results.

LEMMA 1. *Suppose the following are given:*

(i) *The hypotheses of Theorem* 1 *are satisfied.*

(ii) $\{(U_i, U_i \cap f(N^k)), i=1, \cdots, q\}$ *is a finite open cover of* $f(N^k)$ *obtained by applying the definition of locally flat.*

(iii) $\epsilon > 0$.

Then there exists an integer r *such that if* α *is a* k*-simplex of* $(N^k)^r$ *and if* $C_{f(\alpha)} = f(C_\alpha)$,

(a) $f(\alpha) \subset C_{f(\alpha)} \subset U_j \cap f(N^k)$ *for some* j.

(b) $C_{f(\alpha)}$ *is* ϵ*-tame in* N^n.

Conclusion (a) is obvious since every open cover of a compact metric space has a Lebesgue number and the limit of the mesh of $f(N^k)^i$ as i approaches infinity is zero.

Let r and j be integers such that conclusion (a) is true. Let h_j be the homeomorphism of $(U_j, U_j \cap f(N^k))$ onto (R^n, R^k). Since $\dot{C}_\alpha$ is a bicollared [1] $k-1$ sphere in N^k, $h_j(f(\dot{C}_\alpha))$ is a bicollared $k-1$ sphere in R^k. Hence $h_j(C_{f(\alpha)})$ is a tame k-cell in R^k and therefore U_j can be triangulated as a combinatorial n-manifold in such a way that $f: C_\alpha \rightarrow U_j$ is a piecewise linear embedding.

We now apply Homma's theorem. Let $M^n = N^n$, $\hat{M}^n =$ a closed regular neighborhood of $C_{f(\alpha)}$ in U_j and $\hat{P}^k = C_{f(\alpha)}$. Homma's theorem asserts that $C_{f(\alpha)}$ is ϵ-tame in N^n.

PROOF OF THEOREM 1. Let r be an integer such that if α is a k-simplex of $(N^k)^r$, Lemma 1 is valid. Let A_i denote the proposition that if K_i is a connected homogeneous k-subcomplex of $(N^k)^r$ containing i k-simplexes, then $f(K_i)$ is ϵ-tame in N^n for each $\epsilon > 0$. It suffices to show that A_i is true for each positive integer i.

A_1 is true by Lemma 1. Suppose A_i is true for $1 \leqq i \leqq j$. Let K_{j+1} be a connected homogeneous k-subcomplex of $(N^k)^r$ containing $j+1$ k-simplexes. Then $K_{j+1} = K_j \cup \alpha$, where K_j is a connected homogeneous k-subcomplex of $(N^k)^r$ containing j k-simplexes and α is a k-simplex of $(N^k)^r$. Let $\epsilon > 0$ and $\epsilon' = \epsilon/2$, then by assumption, $f(K_j)$ is ϵ'-tame in N^n and by Lemma 1, $C_{f(\alpha)}$ is ϵ'-tame in N^n.

Let h_k and h_α be the ϵ'-homeomorphisms for $f(K_j)$ and $C_{f(\alpha)}$ respectively such that they are ϵ'-tame in N^n. Let U_α be an open ball neighborhood of $h_\alpha(C_{f(\alpha)})$ in N^n, and $W_\alpha = h_\alpha^{-1}(U_\alpha)$.

We will complete the proof of A_{j+1} by applying Gluck's modification of Homma's theorem. Let $M^n = h_k(w_\alpha)$ triangulated as an open subset of N^n, $\hat{M}^n = h_k(W_\alpha)$ triangulated as a combinatorial n-manifold such that $h_k f: C_\alpha \rightarrow h_k(W_\alpha)$ is a piecewise linear embedding. Take $\hat{P}^k = h_k[C_{f(\alpha)} \cap f(K_j)] \cup h_k(f(\alpha))$ and $\hat{L} = h_k[C_{f(\alpha)} \cap f(K_j)]$. By choice of h_k, $\hat{L}$ is piecewise linearly embedded in both M^n and $\hat{M}^n$. Let ϵ'' be picked such that $0 < \epsilon'' < \epsilon'$ and such that $[U_{\epsilon''}(h_k(f(\alpha)))] \cap h_k(f(K_j)) \subset \hat{L}$

and $\text{Cl}[U_{\epsilon''}(h_k(f(\alpha)))] \subset h_k(W_\alpha)$. The hypotheses of Gluck's theorem are satisfied, hence there exists an ϵ''-homeomorphism g of M^n onto itself such that $\hat{P}^k - \hat{L}$ is ϵ''-tame in M^n under g and $g|\hat{L} = 1$. g, which is the identity on $h_k[f(K_j) \cap W_\alpha]$ and near the boundary of $h_k(W_\alpha)$, may be extended via the identity to an ϵ''-homeomorphism $\bar{g}$ of N^n onto itself.

Then $F = \bar{g}h_k$ is an ϵ-homeomorphism of N^n onto itself, such that under F, $f(K_{j+1})$ is ϵ-tame in N^n. Thus A_{j+1} is true and by induction the theorem is proved.

Theorems 1 and 4 reduce the proof of Theorem 2 to the piecewise linear case which has already been handled in [7].

The proof of Theorem 3 is an easy application of Homma's theorem. The following lemma also follows from Homma's theorem.

LEMMA 2. *Suppose the following are given:*

(i) *The hypotheses of Theorem 4 are satisfied except P^k is finite.*

(ii) $\{(U_\lambda, U_\lambda \cap f(P^k)), \lambda = 1, \cdots, q\}$ *is a finite open cover of* $f(P^k)$ *obtained by applying Definition 2 of locally tame.*

(iii) $\epsilon > 0$.

Then there exists a triangulation of $f(P^k)$ such that the closed simplicial neighborhood of any simplex in this triangulation of $f(P^k)$ is contained in $U_j \cap f(P^k)$ for some j and is ϵ-tame in N^n.

Lemma 2, together with Gluck's modification of Homma's theorem are sufficient to prove Theorem 4.

Actually, Lemma 1 shows that locally flat closed combinatorial manifolds with the correct codimension are locally tame according to Definition 2. This, together with Theorem 4, would yield Theorem 1 as a corollary.

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