9. D. Ray, *On spectra of second order differential operators*, Trans. Amer. Math. Soc. **77** (1954), 299–321.
10. F. Riesz and B. Sz.-Nagy, *Functional analysis*, trans. from 2nd French edition by L. Boron, Ungar, New York, 1955.
11. F. Rellich, *Perturbation theory of eigenvalue problems*, Lecture Notes, New York University, New York, 1955.
12. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Part II, Oxford Univ. Press, New York, 1958.

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# DISTRIBUTION MODULO 1 AND SETS OF UNIQUENESS

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Communicated by A. Zygmund, July 6, 1963

A linear set $E\subset(0, 1)$ is said to be a set of uniqueness (set $U$) for trigonometric expansion if no trigonometric series exists (except vanishing identically) which converges to zero in the set $CE$ complementary to $E$. Following Nina Bary we shall say that $E$ is a set of uniqueness "in the wide sense" (set $U^*$) if no Fourier-Stieltjes series exists (except vanishing identically) which converges to zero in $CE$. If $E$ is a *closed* set $U^*$ it means (see [1, Vol. 1, pp. 344–359, Vol. 2, p. 160]) that $E$ does not carry any measure whose Fourier-Stieltjes coefficients tend to zero. If $E$ is a *closed* set $U$ (i.e. of uniqueness "strict sense") it means that $E$ does not carry any measure or *pseudo-measure* (cf. [2]) with coefficients tending to zero.

DEFINITION. A real sequence of numbers $\{u_\kappa\}_1^\infty$ will be said to be "badly distributed" modulo 1 if there exists at least one characteristic function $X(x)$ of open interval $\Delta\subset(0, 1)$ periodic with period 1 such that

$$\limsup_{\kappa=\infty}\frac{X(u_1)+\cdots+X(u_\kappa)}{\kappa}<\int_0^1 X(x)dx=|\Delta|$$

when $|\Delta|$ stands for the length of $\Delta$.[1]

REMARK. It is easy to see that under this hypothesis there exists a $\Delta$ with rational end-points having the same property.

THEOREM. *Let* $E\subset(0, 1)$ *be a linear set such that there exists an infinite sequence of positive integers* $\{n_\kappa\}_1^\infty$ *increasing to infinity, with the*

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[1] The reader will convince himself that all the argument which follows is applicable in the case we suppose lim inf$>\Delta$.

*property that for every* $x \in E$, *the sequence* $\{n_\kappa x\}$ *is badly distributed modulo* 1. *Then* $E$ *is a set of the type* $U^*$.

We shall make use of the three following known lemmas:

LEMMA I (*see* [1, *Vol.* 2, *pp.* 145, 160]). *In order to prove that a closed set does not carry a nonvanishing measure with Fourier-Stieltjes coefficients tending to zero, it is sufficient to prove that it does not carry a* positive *measure having this property.*

LEMMA II (*see* [1, *Vol.* 2, *p.* 144]). *Let*

$$d\mu \sim \sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$$

*be a Fourier-Stieltjes series and let*

$$X(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{2\pi i n x}$$

*be the Fourier series of the characteristic function* $X(x)$ *of an interval* $\Delta \subset (0, 1)$. *Then if the Fourier-Stieltjes coefficients* $c_n$ *tend to zero as* $n \to \infty$, *one has*

$$\lim_{m=\infty} \int_0^1 X(mx)d\mu = c_0\gamma_0 = \int_0^1 X(x)dx \cdot \int_0^1 d\mu(x).$$

LEMMA III (*see* [1, *Vol.* 2, *p.* 160]). *A set* $E$ *which is the union of a denumerable infinity of* closed *sets* $F_n$ *each of which is of the type* $U^*$, *is also of the type* $U^*$.

PROOF OF THE THEOREM. Taking into account the remark following the definition of "bad distribution," we see that to every $x \in E$ corresponds a characteristic function of open interval with rational endpoints, thus belonging to a denumerable family $\{X_m(x)\}_1^\infty$. Let $E_m$ be the subset of points $x$ of $E$ corresponding to the same function $X_m(x)$. $E$ is the union of all the sets $E_m$.

The set $E_m$ is itself the union of sets $E_{m,h}$ ($h$ a positive integer $1 \leq h < \infty$) such that

$$\frac{X_m(n_1 x) + \cdots + X_m(n_\kappa x)}{\kappa} < \int_0^1 X_m(x)dx \quad \text{for} \qquad \kappa \geqq h.$$

The set $E_{m,h}$ is in turn the union of sets $E_{m,h,s}$, where

$$\text{(1)} \qquad \frac{X_m(n_1 x) + \cdots + X_m(n_\kappa x)}{\kappa} \leqq \int_0^1 X_m(x)dx - \frac{1}{s} \qquad (\kappa \geqq h)$$

where $s$ takes all positive integral values.

Since $X_m(x)$ is lower-semicontinuous, the set $E_{m,h,s}$ is *closed*. We shall show that it is of the type $U^*$. Suppose, in fact, that it carries a *positive* (see Lemma I) nonvanishing measure $d\mu$ with Fourier-Stieltjes coefficients tending to zero. Multiplying (1) by $d\mu$ and integrating with respect to $d\mu$ we set

$$\frac{\int X_m(n_1x)d\mu + \cdots + \int X_m(n_\kappa x)d\mu}{\kappa}$$

$$\leqq \left(\int_0^1 X_m dx\right)\left(\int_0^1 d\mu(x)\right) - \frac{\int_0^1 d\mu}{s} \qquad (\kappa \geqq h).$$

Since for $\kappa \to \infty$ the first member tends (Lemma II) to $\int U_m dx \cdot \int_0^1 d\mu(x)$, this leads to a contradiction, and $E_{m,h,s}$ is a $U^*$ set.

It is now enough to use Lemma III to prove the theorem since $E$ is the union of the denumerable family $E_{m,h,s}$ ($m$, $h$, $s$ positive integers).

APPLICATION. Consider the set of numbers in (0, 1) written in the dyadic system $x = \epsilon_1/2 + \cdots + \epsilon_\kappa/2^\kappa + \cdots$ ($\epsilon_\kappa = 0, 1$) having the property that

$$\limsup \frac{\epsilon_1 + \cdots + \epsilon_\kappa}{\kappa} < \frac{1}{2} \cdot$$

This set is of the type $U^*$. This is an immediate consequence of our theorem if we remark that $\epsilon_\kappa = X(2^\kappa x)$ when $X(x)$ is the characteristic function of the interval (1/2, 1). (Here the family $X_m$ is reduced to a single interval.[2])

## BIBLIOGRAPHY

1. A. Zygmund, *Trigonometric series*, 2nd ed., Cambridge Univ. Press, 1959.

2. J.-P. Kahane and R. Salem, *Sur les ensembles linéaires ne portant pas de pseudo-mésures*, C. R. Acad. Sci. Paris **243** (1956), 1185–1187.

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[2] *Added in proof*. The application given here is nothing but the easy part of an important paper of I. I. Pyatetski-Shapiro (Moskov. Gos. Univ. Uč. Zap. **165**, Mathematika **7** (1954), 79–97), where he constructs a set $U^*$ which is not a set $U$. It suggests the following question, that the authors were not able to solve: is the set of non-normal numbers $x$ (i.e., numbers $x$ such that $\limsup (\epsilon_1 + \cdots + \epsilon_K)/K > 1/2$ or $\liminf < 1/2$) a set $U^*$? In other words, is it a set of measure zero with respect to every positive measure whose Fourier coefficients tend to zero at infinity?