VARIATION DIMINISHING TRANSFORMATIONS

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The present note is a brief summary of the principal results of the author's doctoral dissertation at Harvard University on an inversion theory and a corresponding representation theory for convolutions of variation diminishing Hankel kernels. These results are parallel to and partly include the theory developed by Hirschman and Widder in [1]. Complete details with proofs will appear in a later publication.

Let γ be a fixed positive number and let

$$\mu(x) = \frac{x^{2\gamma+1}}{2^{\gamma+1/2}\Gamma(\gamma + 3/2)}.$$

Denote by $L^p(0, \infty; d\mu)$, $1 \leqq p < \infty$, the space of all real-valued measurable functions f defined on $(0, \infty)$ for which the norm $\|f\|_p$ is finite where

$$\|f\|_p = \left[\int_0^\infty |f(x)|^p d\mu(x)\right]^{1/p}.$$

$L^\infty(0, \infty; d\mu)$ denotes the space of those functions f for which $\|f\|_\infty$ is finite where

$$\|f\|_\infty = \operatorname*{ess.\,l.u.b.}_{0<x<\infty} |f(x)|.$$

Define

$$\mathcal{J}(x) = 2^{\gamma-1/2}\Gamma(\gamma + 1/2)x^{1/2-\gamma}J_{\gamma-1/2}(x)$$

and

$$\mathcal{I}(x) = 2^{\gamma-1/2}\Gamma(\gamma + 1/2)x^{1/2-\gamma}I_{\gamma-1/2}(x),$$

where $J_{\gamma-1/2}(x)$ is the ordinary Bessel function and $I_{\gamma-1/2}(x)$, the Bessel function of imaginary argument. The Hankel transform $f^\frown(x)$ of a function f of L^1 is given by $f^\frown(x) = \int_0^\infty \mathcal{J}(xt)f(t)d\mu(t)$, $0 \leqq x < \infty$. Let

$$(1) \qquad G_N(x) = \int_0^\infty \frac{\mathcal{J}(xt)}{E_N(t)} d\mu(t), \qquad 0 \leqq x < \infty, N = 0, 1, 2, \cdots,$$

where

$$E_N(t) = \prod_{k=N+1}^{\infty}\left(1+\frac{t^2}{a_k^2}\right), \qquad N = 0, 1, 2, \cdots,$$

with $0<a_1\leqq a_2\leqq \cdots$ and $\sum_{k=1}^{\infty} 1/a_k^2<\infty$. $G_0(x)$ is written simply as $G(x)$. Let $\Delta(x, y, z)$ be the area of a triangle with sides x, y, z if such a triangle exists. Let

$$D(x, y, z) = \frac{2^{3\gamma-5/2}[\Gamma(\gamma+1/2)]^2}{\Gamma(\gamma)\pi^{1/2}}(xyz)^{-2\gamma+1}[\Delta(x, y, z)]^{2\gamma-2},$$

if $\Delta(x, y, z)$ exists and zero otherwise. Define the associated function $f(x, y)$ of a function $f(x)$ of L^1 by

$$f(x, y) = \int_0^{\infty} f(u)D(x, y, u)d\mu(u), \qquad 0 \leqq x, y < \infty.$$

It can be shown that the function $G_N(x, y)$ associated with $G_N(x)$ is given by

$$G_N(x, y) = \int_0^{\infty} \frac{\mathcal{G}(xt)\mathcal{G}(yt)}{E_N(t)}\,d\mu(t), \qquad 0 \leqq x, y < \infty. \tag{2}$$

For any two functions f and g of L^1, let

$$f \# g(x) = \int_0^{\infty} f(x, y)g(y)d\mu(y), \qquad 0 \leqq x < \infty.$$

It follows that the space L^1, with multiplication defined by #, forms a Banach algebra. For the particular convolution $G \# \phi(x)$ the following result holds:

CONVERGENCE THEOREM. *Let ϕ be a function integrable in any finite interval, and let*

$$\int_0^{\infty} G(x_0, y)\phi(y)d\mu(y) = \lim_{T\to\infty}\int_0^{T} G(x_0, y)\phi(y)d\mu(y), \qquad x_0 \geqq 0,$$

converge conditionally. Then

$$\int_0^{\infty} G(x, y)\phi(y)d\mu(y)$$

converges conditionally for all x and uniformly for x in any finite interval. In this respect $G \# \phi(x)$ behaves like the familiar Stieltjes transform.

If H is a real-valued function of L^1 and ϕ, a real-valued, continuous function of L^∞, then H is said to be a variation diminishing #-kernel if and only if, for every such ϕ, the number of variations of sign of $H \# \phi$ does not exceed the number of variations of sign of ϕ. In [2], Hirschman generalized an earlier result of I. J. Schoenberg by proving that a #-kernel G is variation diminishing if and only if it has the form (1).

The main inversion theorem under the least restrictive hypotheses is the following:

INVERSION THEOREM. *Let ϕ be a function integrable on every finite interval and let*

$$f(x) = \int_0^\infty G(x, t)\phi(t)d\mu(t), \qquad 0 < x < \infty,$$

converge conditionally. Then

$$\lim_{N\to\infty} \prod_{k=1}^{N} \left(1 - \frac{\Delta_x}{a_k^2}\right) f(x) = \phi(x)$$

where

$$\Delta_x h(x) = h''(x) + \frac{2\gamma}{x} h'(x)$$

if

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} [\phi(t) - \phi(x)]d\mu(t) = 0,$$

a condition which holds almost everywhere.

The behavior of the variation diminishing kernels $G(x, y)$ and of the quotients $G_N(x, y)/G(x, y)$ of kernels plays a central role in the development of the theory. For example, the matrix $[G(x_i, y_j)]_{1\leq i,j\leq n}$ is totally non-negative, where a real matrix is said to be totally non-negative if and only if all its minors of any order are non-negative. From this it follows that $G(x, y)/G(x_0, y)$ is a monotonic decreasing function of y for $x_0 > x$ and monotonic increasing for $x_0 < x$. The additional fact that

$$\frac{G(x, y)}{G(x_0, y)} \sim \frac{\mathcal{J}(a_1 x)}{\mathcal{J}(a_1 x_0)}, \qquad y \to \infty,$$

which follows from (2) by an appeal to the calculus of residues, leads to the proof of the convergence theorem stated earlier.

The corresponding principal representation theorem is the following:

REPRESENTATION THEOREM. *Necessary and sufficient conditions that a function f be given by*

$$f(x) = \int_0^\infty G(x, t)d\psi(t),$$

with $\psi(t) \uparrow$ are that

(i) $f(x) \in C^\infty$, $0 \leqq x < \infty$

(ii) $f^{(2k+1)}(0) = 0$, $k = 0, 1, 2, \cdots$,

(iii) $f(x) = o(\mathcal{G}(a_1 x))$, $x \to \infty$,

(iv) $\prod_{k=1}^{N_i} (1 - \Delta_x/a_k^2) f(x) \geqq 0$, $0 < x < \infty$, $1 = N_0 < N_1 < \cdots$.

The proof of this result depends on a fundamental representation theorem which states that a function f satisfying conditions (i)–(iv) above is given by

$$f(x) = \int_0^\infty G_{N_i}^*(x, y) \left[\prod_{k=1}^{N_i} \left(1 - \frac{\Delta_y}{a_k^2}\right) f(y) \right] d\mu(y),$$
$$0 < x < \infty, 1 = N_0 < N_1 < \cdots,$$

where

$$G_N^*(x) = \int_0^\infty \frac{\mathcal{G}(xt)}{\prod_{k=1}^{N} \left(1 + \frac{t^2}{a_k^2}\right)} d\mu(t), \qquad 0 < x < \infty, N = 1, 2, \cdots.$$

This basic result in conjunction with an application of Helley's theorem and an appeal to Tauberian theorems serves to establish the main representation theorem.

REFERENCES

1. I. I. Hirschman, Jr. and D. V. Widder, *The convolution transform*, Princeton Univ. Press, Princeton, N. J., 1955.

2. I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, J. Analyse Math. 8 (1960–61), 307–336.

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