ON THE REAL PARTS OF ZEROS OF
EXPONENTIAL POLYNOMIALS

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In the study of linear differential systems with a single time lag we are led to the study of the nature of the zeros of the characteristic exponential polynomial, which are in general infinite in number. In questions of stability the primary problem is determining the nature of the real parts of these zeros, stability only occurring if the real parts are all negative. The most notable achievement in this line is that of Pontrjagin [2] which states that if the exponential polynomial lacks a principal term (see Bellman and Cooke [1, p. 440]), it has an infinite number of zeros with arbitrarily large positive real parts. What happens when there is a principal term has until now not been discussed. This paper proposes to fulfill that need.

In many applications it is possible to vary the coefficients slightly, and, in so doing, alter the zeros slightly. With only a finite number of zeros with positive real parts it may be possible to alter the coefficients in such a way as to achieve stability. If an infinite number of zeros with arbitrarily large positive real parts occurs, any hope of stabilizing the system vanishes.

We will consider a linear differential system with time lag τ having as its characteristic equation

$$(1) \quad F(z) = z^n + az^{n-1} + \cdots - Ke^{i\theta}e^{-\tau z}(z^m + bz^{m-1} + \cdots) = 0$$

where $\tau > 0$, $K \geq 0$ and $\theta \geq 0$ are real constants and a and b are complex constants.

THEOREM I. *If $n > m$: the number of zeros of $F(z)$ with positive real part (or lying in any right half plane) is finite; if $K \neq 0$, $F(z)$ has an infinite number of zeros with arbitrarily large negative real part.*

II. *If $n = m$: when $K \neq 0$ $F(z)$ has an infinite number of zeros given by*

$$(2) \qquad\qquad (1/\tau)(\log K + i(\theta + 2k\pi)) + o(1)$$

as $k = 0, \pm 1, \pm 2, \cdots$, and only a finite number of other zeros. If $K < 1$, $F(z)$ has only a finite number of zeros with positive real part. If $K > 1$, $F(z)$ has only a finite number of zeros with negative real part.

III. *If $n < m$: the number of zeros of $F(z)$ with negative real part (or lying in any left half plane) is finite; if $K \neq 0$, $F(z)$ has an infinite number of zeros with arbitrarily large positive real parts.*

PROOF. Clearly III follows from I by replacing z by $-z$ and multiplying by $e^{-\tau z}$.

I. Choose any real number a. Suppose $F(z)$ has an infinite number of zeros with real part greater than a. These must approach infinity since $F(z)$ is an entire function. Since $|e^{-\tau z}|$ is bounded by $e^{-\tau a}$ to the right of a, we have $F(z) = z^n(1 + o(1))$. Thus $F(z)$ is asymptotic to z^n giving a contradiction. The proof of the existence of an infinite number of zeros of $F(z)$ with arbitrarily large negative real part is due to Pontrjagin [2].

II. As in I, choose any real a. $|e^{-\tau z}|$ is then bounded to the right of a by $e^{-\tau a}$, and $F(z) = z^n(1 - Ke^{i\theta}e^{-\tau z} + O(1/|z|))$. By use of Rouché's Theorem we see that the zeros of $F(z)$ become arbitrarily close to the zeros of $1 - Ke^{i\theta}e^{-\tau z}$ and there can be none elsewhere. Hence these zeros have the form (2).

Clearly these asymptotic zeros ultimately lie on the same side of $x = 0$ as log K, i.e., in the right half plane if $K > 1$ and in the left half plane if $K < 1$.

To consider zeros in any left half plane, replace z by $-z$ and multiply by $e^{-\tau z}$. In so doing we find the result is the same as before.

In many situations, for example, feed back systems, K is a parameter which can be chosen more or less arbitrarily. If $\tau = 0$, there are many well-known tests to determine stability as well as to determine which values of K lead to stable systems. At present these techniques are lacking for differential-difference systems, at least in French, German or English translations. It is hoped that the absence of suitable techniques will soon be remedied.

<div align="center">REFERENCES</div>

1. Richard Bellman and Kenneth L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.

2. L. S. Pontrjagin, *On the zeros of some elementary transcendental functions*, Amer. Math. Soc. Transl. (2) 1 (1955), 95–110.

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