THE GROUP OF HOMOTOPY EQUIVALENCES OF A SPACE[1]

BY M. ARKOWITZ AND C. R. CURJEL

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1. **Introduction.** Let $\mathcal{E}(X)$ denote the collection of homotopy classes of homotopy equivalences of a space X with itself. Composition of maps induces a group structure in $\mathcal{E}(X)$. From the point of view of categories $\mathcal{E}(X)$ is the group of equivalences of the object X in the category of spaces and homotopy classes of maps. Thus it is the homotopy analog of the automorphism group of a group and the group of homeomorphisms of a space.

In this note we present some theorems which relate properties of the homotopy groups of X to algebraic properties of $\mathcal{E}(X)$. In such a study one encounters the difficulties associated with the problem of composing homotopy classes of maps. In addition one can easily show that for any finite group T there exists a finite complex X such that $\mathcal{E}(X)$ contains T as a subgroup. Thus any group theoretic property which does not hold for all finite groups cannot be true for all groups $\mathcal{E}(X)$.

The hypotheses of all of our theorems are not intricate, and thus our results provide specific information on $\mathcal{E}(X)$ for many X. §2 contains theorems on the group of equivalences of any 1-connected finite complex, and §3 deals with associative H-spaces. At the end of each section we give a brief description of our methods. Details and applications will appear elsewhere.

Various results on $\mathcal{E}(X)$ have been obtained by Barcus-Barratt [**1**, §6], P. Olum (to appear) and D. W. Kahn (to appear). Furthermore, W. Shih [**5**] has constructed a spectral sequence for $\mathcal{E}(X)$.

We should like to thank R. P. Langlands for several discussions on Proposition 9.

2. **General theorems.** We consider only 1-connected spaces of the homotopy type of a CW-complex with finitely generated homotopy groups in all dimensions. Let $X^{(n)}$ be an nth Postnikov section of X. A straightforward obstruction argument yields

LEMMA 1. *If X is a finite* CW-*complex then* $\mathcal{E}(X)\approx\mathcal{E}(X^{(n)})$ *for all* $n>\dim X$.

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By assigning to an element of $\mathcal{E}(X)$ its induced homotopy automorphism, one obtains a homomorphism $I: \mathcal{E}(X) \to \prod_k \operatorname{Aut} \pi_k(X)$ whose kernel is denoted by $\mathcal{E}_{\#}(X)$.

PROPOSITION 2. *There is an exact sequence*

$$T_n(X) \to \mathcal{E}_{\#}(X^{(n)}) \to \mathcal{E}_{\#}(X^{(n-1)}),$$

where $T_n(X)$ *denotes the kernel of the homomorphism* $H^n(X; \pi_n(X)) \to \operatorname{Hom}(\pi_n(X), \pi_n(X))$.

Note that the map $T_n(X) \to \mathcal{E}_{\#}(X^{(n)})$ carries the sum of two cohomology classes into the composition of two homotopy equivalences.

THEOREM 3. *Let X be a finite* CW-*complex such that rank* $\pi_i(X) \leqq 1$ *for all* $i \leqq \dim X+1$. *Then* $\mathcal{E}(X)$ *satisfies the maximal condition* (*i.e.*, $\mathcal{E}(X)$ *and all its subgroups are finitely generated*).

Examples of spaces X such that $\mathcal{E}(X)$ does not satisfy the maximal condition can, for instance, be obtained from Corollary 8.

Now let $|T|$ stand for the order of the group T and $h_i: \pi_i(X) \to H_i(X)$ for the Hurewicz homomorphism.

THEOREM 4. *Let X be as in Theorem* 3. *Then* $\mathcal{E}(X)$ *is finite if the group* $\operatorname{Hom}(\operatorname{coker} h_i, \pi_i)$ *is finite for all* i; *in this case*

$$|\mathcal{E}(X)| \leqq \prod_{i=2}^{\dim X+1} p_i \,|\operatorname{Hom}(\operatorname{coker} h_i, \pi_i)|\, |\operatorname{Ext}(H_{i-1}, \pi_i)|$$

where $\pi_i = \pi_i(X)$, $H_i = H_i(X)$ *and* $p_i = |\operatorname{Aut} \pi_i|$.

As a group defined by composition of maps $\mathcal{E}(X)$ is generally non-abelian. Theorem 5 gives conditions for $\mathcal{E}(X)$ to be solvable. If p is a prime we denote by $n(H, p^k)$ the number of times the cyclic group Z_{p^k} occurs in the canonical decomposition of the finitely generated abelian group H.

THEOREM 5. *Let X be a finite* CW-*complex such that*

(i) *rank* $\pi_i(X) \leqq 1$,

(ii) *for any* k

$$n(\pi_i(X), p^k) \leqq 2 \quad \text{if } p = 2, 3,$$
$$\leqq 1 \quad \text{otherwise}$$

for all $i \leqq \dim X+1$. *Then* $\mathcal{E}(X)$ *is a solvable group.*

The proofs of Theorems 3, 4 and 5 proceed similarly. By Lemma 1 an N-dimensional complex X can be replaced by its Postnikov section $Y = X^{(N+1)}$. Then in the exact sequence

$$1 \to \mathcal{E}_{\#}(Y) \to \mathcal{E}(Y) \xrightarrow{I} \sum_k \operatorname{Aut} \pi_k(Y)$$

the group on the right is a finite direct sum. Now one studies $\mathcal{E}_{\#}(Y)$ and $\sum_k \operatorname{Aut} \pi_k(Y)$ separately. The group $\sum_k \operatorname{Aut} \pi_k(Y)$ is dealt with purely algebraically. On the other hand it follows from Proposition 2 that $\mathcal{E}_{\#}(Y)$ possesses all properties which are shared by finitely generated abelian groups and which carry over to subgroups, factor groups and extensions. This observation immediately yields Theorem 3. The condition rank $\pi_i(X) \leqq 1$ of Theorem 4 implies that $\mathcal{E}(Y)$ is finite if and only if $\mathcal{E}_{\#}(Y)$ is finite. By Proposition 2 the latter group is finite if Hom(coker h_i, π_i) is finite. The assumptions in Theorem 5 together with an argument based on [6, Satz 8] imply that $\sum_k \operatorname{Aut} \pi_k(Y)$ is solvable.

REMARKS. (a) Clearly Theorems 3, 4 and 5 also hold for a space with finitely many homotopy groups.

(b) It is easily seen that for a finite complex X the group $\mathcal{E}_{\#}(X)$ is a subgroup of $\mathcal{E}_{\#}(X^{(n)})$ for $n > \dim X$. Thus, for instance, $\mathcal{E}_{\#}(X)$ always is solvable and satisfies the maximal condition.

(c) There are examples of 1-connected finite complexes which show that each of the hypotheses in Theorems 3, 4 and 5 is necessary.

3. **Theorems for H-spaces.** In this section G denotes a 1-connected associative H-space of the homotopy type of a finite CW-complex. Let $n_1, \cdots, n_k$ stand for the dimensions of the algebra generators of the cohomology algebra of G with rational coefficients.

THEOREM 6. (a) $\mathcal{E}(G)$ *is finitely generated.*[2] (b) $\mathcal{E}(G)$ *is a finite group if and only if the n_ith Betti number of G equals one for all* $i = 1, \cdots, k$.

The following theorem shows that for certain G the group $\mathcal{E}(G)$ contains free subgroups of any rank.

THEOREM 7. $\mathcal{E}(G)$ *contains a nonabelian free subgroup on at least two generators if and only if rank* $\pi_i(G) > 1$ *for some* i.

COROLLARY 8. $\mathcal{E}(G)$ *satisfies the maximal condition if and only if rank* $\pi_i(G) \leqq 1$ *for all* i.

The proofs of Theorems 6 and 7 rely on the following considerations. First of all, the collection $\pi(G, G)$ of homotopy classes of basepoint preserving maps from G into itself has the structure of a near-

[2] W. Shih informs us that he has proved that $\mathcal{E}(Y)$ is finitely generated if Y has finitely many homotopy groups. By our Lemma 1 this implies that $\mathcal{E}(X)$ is finitely generated for any 1-connected finite complex X.

ring [2]. Secondly, it can be shown that the homomorphism $\pi(G^{(n)}, G^{(n)}) \to \sum_{k \leq n} \mathrm{Hom}(\pi_k(G), \pi_k(G))$ is an epimorphism modulo the class of finite abelian groups. In addition in the proof of Theorem 7 one needs the result of Frasch [4, §6] that the principal congruence group mod p of two-by-two integer matrices is a free group on at least two generators. An important step in the proof of Theorem 6(a) is the application of the following proposition to the *ring* $\pi(G^{(n)}, G^{(n)})/\mathrm{Ker}\ \Omega$ where $\Omega: \pi(G^{(n)}, G^{(n)}) \to \pi(\Omega G^{(n)}, \Omega G^{(n)})$ is the loop homomorphism.

PROPOSITION 9. *Let A be an associative ring with* 1 *whose additive group is finitely generated. Then the group of units of A is also finitely generated.*

This proposition is a consequence of a result of Borel and Harish-Chandra [3, Theorem 6.12].

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PRINCETON UNIVERSITY,
THE INSTITUTE FOR ADVANCED STUDY AND
CORNELL UNIVERSITY