A STRICT MAXIMUM THEOREM FOR ONE-PART FUNCTION SPACES AND ALGEBRAS

BY H. S. BEAR[1]

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The idea of a "part" of the spectrum of a function algebra was introduced by Gleason in [3]. Recently Bishop [2] extended one of the results of [3] and showed that any two points in one part of a function algebra have representing measures which are mutually absolutely continuous. Motivated by Bishop's work, we have shown [1] that the concept of a part extends to arbitrary linear spaces of functions in a way which generalizes the idea for an algebra. Our purpose here is to show how this concept can be used to prove a strict maximum theorem for subalgebras or subspaces.

Let A be a separating, uniformly closed subalgebra of $C_c(X)$, all continuous complex-valued functions on a compact space X, and let S_A denote the space of nonzero homomorphisms of A. Then S_A is a compact subset of A^* with the w^* topology, and X is homeomorphically embedded in S_A as the set of evaluation functionals. The functions in A are generally regarded as being extended from X to continuous functions on all of S_A. If A^* is given the norm topology rather than the customary w^* topology, then the relation $x \sim y$ ($x, y \in S_A$) defined by $\|x-y\| < 2$ is an equivalence relation on S_A, and the equivalence classes are called the Gleason parts of S_A.

Now let $C_r(X)$ be all continuous real functions on X, and let B be a separating subspace of $C_r(X)$ which contains the constants. The parts of X (with respect to B) are defined to be the equivalence classes under the relation $x \sim y$ defined by the condition that there exist a real number $a > 1$ such that $1/a < u(x)/u(y) < a$ for all positive functions $u \in B$. In [1] it is shown that if $B = \mathrm{Re}\, A$ for a function algebra A, then the B-parts coincide with the A-parts. We obtain a geometric interpretation of a part by regarding X as embedded in B^*, and representing B as the weak dual of B^*, restricted to the closed convex hull T_B of X in B^*. The parts of T_B, regarding B as a subspace of $C_r(T_B)$, are (see [1]) those (necessarily convex) subsets Π of T_B which do not contain any extreme points of T_B, and have the property that if $x, y \in \Pi$, then the line in B^* joining x and y intersects T_B in a segment containing x and y in its interior. The parts of

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X, considered as a subset of T_B, are the intersections of X with the parts of T_B.

We can now state our maximum theorem.

THEOREM. *If B is a separating subspace of $C_r(X)$ containing the constants, Γ is the Silov boundary of B in X, and $X \sim \Gamma$ is a single part, then no function in B assumes its maximum or minimum at a point of $X \sim \Gamma$ unless constant on $X \sim \Gamma$.*

PROOF. Let us consider X as embedded in B^* and let $T_B = \langle X \rangle$. Then B is isometrically isomorphic to the continuous linear functionals on B^*, restricted to T_B, and Γ is the closure of the extreme points of T_B. Our hypothesis is that $\Delta = X \sim \Gamma$ is contained in a single part Π of T_B. Let u be a nonconstant function in B, with $u(x) \neq u(y)$ and x, $y \in \Delta$. Then the line joining the functionals x and y of B^* extends beyond x and y in Π. This segment need not (probably does not) intersect X other than at x and y. However, u is linear on T_B, so $u(z) > u(x)$ for some $z \in \Pi$ on the line joining x and y. The maximum of the linear functional u on the convex set T_B will be assumed at an extreme point $\theta \in \Gamma$, and hence $\|u\| = u(\theta) \geqq u(z) > u(x)$. Since x is any point of Δ, the theorem is proved.

If A is a function algebra and some $f \in A$ assumes its maximum modulus at x, then some $u \in \operatorname{Re} A$ assumes its maximum at x. Hence we have the following corollary.

COROLLARY. *If A is a function algebra, Γ is the Silov boundary of A in its spectrum S_A, and $S_A \sim \Gamma$ is one part, then no function in A assumes its maximum modulus at a point of $S_A \sim \Gamma$ unless constant on $S_A \sim \Gamma$.*

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA