THE HAUSDORFF DIMENSION OF SINGULAR SETS OF PROPERLY DISCONTINUOUS GROUPS IN N-DIMENSIONAL SPACE

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1. **Hausdorff dimension.** Suppose E is a compact subset of N-dimensional euclidean space, E^N. We denote by $m_\alpha(E)$ the Hausdorff α-dimensional measure of E and by $d(E)$ the Hausdorff dimension of E, i.e. the unique non-negative number such that

$$m_\alpha(E) = 0 \qquad \text{for } \alpha > d(E)$$

and

$$m_\alpha(E) = +\infty \quad \text{for } 0 \leqq \alpha < d(E).$$

We shall need the following result.

THEOREM A [6]. *Let E be a compact subset of E^2. Then $d(E)>0$ implies E has positive logarithmic capacity.*

2. **Spherical Cantor sets.**

DEFINITION 1 [2], [7]. We say E is a spherical Cantor set if and only if E can be expressed in the form

$$E = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\cdots,i_n=1}^{K} \Delta_{i_1\cdots i_n}$$

where K is a positive integer ($K \geqq 2$) and the $\Delta_{i_1\cdots i_n}$ are closed N-dimensional spheres (of radius $r_{i_1\cdots i_n}$) satisfying

(a) $\Delta_{i_1\cdots i_n} \supset \Delta_{i_1\cdots i_{n+1}}$ $(i_{n+1}=1,\cdots,K)$,

(b) $\Delta_1,\cdots,\Delta_K$ are mutually disjoint,

(c) there exists a constant A, $1>A>0$, such that

$$r_{i_1\cdots i_n i_{n+1}} \geqq A r_{i_1\cdots i_n} \qquad (i_{n+1} = 1,\cdots,K)$$

and

(d) there exists a constant B, $1>B>0$, such that

$$\rho(\Delta_{i_1\cdots i_n s}, \Delta_{i_1\cdots i_n t}) \geqq B r_{i_1\cdots i_n} \qquad (s, t = 1,\cdots,K;\, s \neq t)$$

where

$$\rho(S, T) = \inf\{\, |s-t|\,;\, s \in S, t \in T\}.$$

We quote the following results.

THEOREM B [2]. *Let E be a spherical Cantor set (in the notation of Definition 1) and suppose that there exist constants $A_1, \cdots, A_K$ $(1 > A_j \geqq 0)$ such that*

$$r_{i_1 \cdots i_n j} \geqq A_j r_{i_1 \cdots i_n} \qquad (j = 1, \cdots, K).$$

Then $d(E)$ satisfies

$$A_1^{d(E)} + \cdots + A_K^{d(E)} \leqq 1.$$

COROLLARY.

$$d(E) \geqq -\frac{\log K}{\log A} > 0$$

and so every spherical Cantor set has positive Hausdorff dimension.

THEOREM C [2]. *There exists a function $F_N(n)$ defined for $n = 2, 3, \cdots$ such that if E is a spherical Cantor set in E^N with K as in Definition 1 then*

$$d(E) \leqq F_N(K) < N.$$

In particular, $d(E) < N$.

3. **Inversion groups.** For $j = 1, \cdots, K+1$ let S_j be an N-dimensional sphere with centre a_j and radius r_j $(r_j > 0)$. Suppose that $S_1, \cdots, S_{K+1}$ are mutually disjoint and for any $x \in E^N$ denote by $I_j(x)$ the inverse point of x with respect to S_j. It is well known that $I_1, \cdots, I_{K+1}$ generate a properly discontinuous group G with the complement of $\bigcup_j S_j$ as a fundamental region [3], [4]. Let E be the singular set of G and define $E_j = E \cap S_j$.

THEOREM 1. *In the above notation E_j is a spherical Cantor set.*

PROOF. We only sketch the proof here.
Define

$$\Delta_{i_1 \cdots i_n} = I_{i_1} \cdots I_{i_{n-1}}(S_{i_n}), \qquad n \geqq 2, \; i_t \neq i_{t+1}$$

and

$$\Delta_{i_1} = S_{i_1}.$$

It is easily seen that

$$E_j = \bigcap_{n=1}^{\infty} \bigcup_{i_1 \cdots i_n = 1}^{K+1} \Delta_{i_1 \cdots i_n} \qquad (i_1 = j, \; i_t \neq i_{t+1})$$

and that (apart from relabelling) this is the form required by Definition 1. Further, conditions (a) and (b) of Definition 1 are trivially true. Since we have

$$\Delta_{i_1\cdots i_n i_{n+1}} \subset \Delta_{i_1\cdots i_n} \subset S_{i_1}$$

we also have for $k \neq i_1$

$$I_k(\Delta_{i_1\cdots i_n i_{n+1}}) \subset I_k(\Delta_{i_1\cdots i_n}) \subset I_k(S_{i_1})$$

and by elementary geometry we can estimate

$$\frac{r_{ki_1\cdots i_n i_{n+1}}}{r_{ki_1\cdots i_n}}$$

in terms of

$$\frac{r_{i_1\cdots i_n i_{n+1}}}{r_{i_1\cdots i_n}}.$$

In this way we obtain

$$\frac{r_{ki_1\cdots i_n i_{n+1}}}{r_{ki_1\cdots i_n}} \geqq \mu(i_1\dots i_n)\frac{r_{i_1\cdots i_n i_{n+1}}}{r_{i_1\cdots i_n}}$$

where $\mu(i_1 \cdots i_n)$ depends only upon $S_1, \cdots, S_{K+1}$ and $r_{i_1\cdots i_n}$.

It is easy to show that there exists $\{\mu_n\}$ such that

$$\mu(i_1 \cdots i_n) \geqq \mu_n > 0$$

and

$$\prod_{n=1}^{\infty} \mu_n = \mu > 0.$$

Thus

$$\frac{r_{i_1\cdots i_n i_{n+1}}}{r_{i_1\cdots i_n}} \geqq \mu > 0$$

and so (c) of Definition 1 is verified. The remaining condition (d) can be similarly verified. We remark here that in the case when $r_1 = \cdots = r_{K+1} = 1$ it can be shown that

$$\frac{r_{i_1\cdots i_n i_{n+1}}}{r_{i_1\cdots i_n}} \geqq \frac{1}{(f^2-1)\prod_{n=0}^{\infty}\left(1+\frac{2}{(e-1)^{2n}}\right)}$$

where

$$e = \min |a_p - a_q|, \qquad f = \max |a_p - a_q| \qquad (p \neq q).$$

A finer estimate in this case gives

$$\frac{r_{i_1 \cdots i_n i_{n+1}}}{r_{i_1 \cdots i_n}} \geqq \frac{C}{|a_{i_n} - a_{i_{n+1}}|^2} \qquad (i_n \neq i_{n+1}),$$

for some positive constant C depending only on $S_1, \cdots, S_{K+1}$.

From Theorems B, C and 1 we can immediately deduce the following results.

THEOREM 2. *Let E be the singular set of some inversion group. Then $d(E) > 0$.*

In particular if $r_1 = \cdots = r_{K+1} = 1$ we have

$$d(E) \geqq \frac{\log K}{2 \log f + \log \prod_{n=0}^{\infty} \left(1 + \frac{2}{(e-1)^{2n}}\right)} > 0.$$

THEOREM 3. *There exists a function $F_N(p)$ defined for $p = 2, 3, \cdots$ such that if E is the singular set of any inversion group with $K+1$ generators then*

$$d(E) \leqq F_N(K) < N.$$

We shall indicate a proof of the following result

THEOREM 4. *There exists a finitely generated inversion group in E^N with singular set E such that $d(E) > N/2$.*

PROOF. Let G_n be the group uniquely determined by the spheres of unit radius and centres $(e_1, \cdots, e_N) \in E^N$ where $e_j = 0, 3, 6, \cdots, 6n$ and let E_n be the singular set of G_n. Clearly,

$$G_1 \subset G_2 \subset \cdots$$

and so

$$E_1 \subset E_2 \subset \cdots.$$

Define

$$d_n = d(E_n)$$

and

$$d_1 \leqq d_2 \leqq \cdots \leqq d_n \leqq \cdots \leqq d = \lim_{n \to \infty} d_n \leqq N.$$

It remains to prove that $d>N/2$, for then $d_n>N/2$ for some finite value of n.

Consider in the construction of E_n a fixed sequence $i_1, \cdots, i_p$. For any $i_{p+1}=1, \cdots, (2n+1)^N$ with $i_p \neq i_{p+1}$ the set of numbers

$$| a_{i_p} - a_{i_{p+1}} |^2$$

always contains the numbers

$$t_1^2 + \cdots + t_N^2 \qquad (t_j = 3, 6, \cdots, 3n).$$

Thus for the constants A_j of Theorem B we can use the n^N numbers

$$\frac{C}{t_1^2 + \cdots + t_N^2} \qquad (t_j = 3, 6, \cdots, 3n)$$

and take the remaining A_j's to be zero. It then follows that

$$\sum_{t_1, \cdots, t_N=1}^{\infty} \frac{1}{(t_1^2 + \cdots + t_N^2)^d}$$

converges and hence $d>N/2$ as required.

4. **Schottky groups.** The definition and details of properly discontinuous groups (and in particular Schottky groups) in the complex plane can be found in [3], [4].

THEOREM D [7, p. 109]. *Let G be a finitely generated Schottky group in the complex plane with singular set E and suppose that $C_1, \cdots, C_{2p}$ are the circles used in the definition of G. Then if E_j is that part of E contained in C_j, E_j is a spherical Cantor set.*

THEOREM E [5]. *Let E be the singular set of a finitely generated Schottky group. Then E has positive logarithmic capacity. It follows that the singular set of any properly discontinuous group has positive logarithmic capacity.*

From Theorems B, C and D we can deduce the following results.

THEOREM 5. *The singular set of any finitely generated Schottky group, and hence of any properly discontinuous group, has positive Hausdorff dimension.*

Theorem A implies that Theorem 5 includes Myrberg's well-known result (Theorem E).

THEOREM 6. *There exists a function $F(p)$ defined for $p=2, 3, \cdots$ such that if E is the singular set of a Schottky group with p generators then*

$$d(E) \leqq F(p) < 2.$$

Let G be a Schottky group generated by $T_1, \cdots, T_p$ such that the mutually external circles C_j, C_{j+p} are the isometric circles of T_j, T_j^{-1} respectively. Suppose further that the radius of C_j is 1 $(j=1, \cdots, 2p)$. Then T_j is equivalent to an inversion in C_j followed by a reflection into C_{j+p} (and possibly a rotation). Since it is only the inversion which alters lengths and since the estimates obtained for the inversion groups were uniform over all possible inversions considered, similar theorems can be proved for Schottky groups as for inversion groups, the proofs requiring little modification.

In particular we have

THEOREM 7. *There exists a finitely generated Schottky group with singular set E such that $d(E)>1$.*

This result has recently been obtained by Akaza [1].

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