INVARIANT FUNCTION ALGEBRAS ON COMPACT SEMISIMPLE LIE GROUPS

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1. Let S^n be the n-sphere, $n>1$. The group G of rotations of n-dimensional Euclidean space acts on S^n. In [1], K. deLeeuw and H. Mirkil studied algebras A of continuous complex valued functions on S^n. Assuming that (i) A contains the constants, (ii) A is uniformly closed, and (iii) A is invariant under G, (i.e. that if $f \in A$ then so does the translate of f by any element of G) they showed that A must be either (i) all continuous functions on S^n, or (ii) just the constants, or (iii) all continuous functions f such that $f(x)=f(x')$ whenever x and x' are antipodal points on S^n. J. Wolf then generalised their result to a wider class of compact connected Riemannian symmetric spaces in [2]. Specifically, Wolf considered the class $\mathcal{C}$ of compact connected Riemannian symmetric spaces X which are *not locally isometric* to a product in which one of the factors is a circle, a group manifold $SU(n)$, $n>2$, $SO(4n+2)$, E_6, or a coset space $SU(n)/SO(n)$, $n>2$, $SU(2n)/Sp(n)$, $n>2$, $SO(4n+2)/SO(2n+1)\times SO(2n+1)$, E_6/F_4, or $E_6/(Sp(4)/\pm I)$.

Here is a short description of Wolf's results. Let X be a compact connected irreducible Riemannian symmetric space. Let G be the component of the identity in the group of isometries $\tilde{G}$ of X. Let Δ be the centralizer of G in $\tilde{G}$. Then Wolf shows that there is a one-one correspondence between uniformly closed G-invariant self-adjoint subalgebras A of $C(X)$ on the one hand, and subgroups Γ of Δ on the other: A being identifiable with the algebra of all f in $C(X)$ such that $f\circ\gamma=f$ for all $\gamma\in\Gamma$. He also showed that if X is in the class $\mathcal{C}$, then every closed G-invariant subspace of $C(X)$ is necessarily self-adjoint. Thus his results give a complete classification of G-invariant closed subalgebras of $C(X)$ containing the constants for X in the class $\mathcal{C}$.

In this note we shall give a short proof of the fact that if X is a compact connected semisimple Lie group and if G is the group $X\times X$ acting on X by two-sided translations, viz. $(u,v): x\rightarrow uxv^{-1}$; $u, v, x\in X$, then any G-invariant closed subalgebra $\neq \{0\}$ of $C(X)$ is necessarily self-adjoint and must contain the constants. This extends the results of Wolf to all compact connected symmetric spaces which are group manifolds of semisimple groups. In particular, spaces of this sort in which one of the factors is locally isometric to $SU(n)$, $n>2$, $SO(4n+2)$

or E_6, are within the scope of our result but are not in the class $\mathcal{C}$ considered by Wolf. Our method is more direct than that of Wolf and it is to be hoped that it may shed some light on the other spaces not of class $\mathcal{C}$ mentioned above. However, our method is simplified by the fact that when the space is a group manifold, the identity component in its group of isometries acts on it by *two-sided* translations. This is not so in general.

2. We shall assume that the reader is familiar with the Peter-Weyl theorem for compact groups and related terminology as given e.g. in Chevalley [3].

Let X be a compact connected group and $C(X)$ be the Banach algebra under pointwise multiplication and the uniform norm, of continuous complex valued functions on X. If G is any group acting on X by $g: x \rightarrow g \cdot x$ and A is a subspace of $C(X)$ we shall say that A is G-invariant if, whenever $f \in A$, so does the function $g \cdot f: x \rightarrow f(g \cdot x)$, $g \in G$. Let Ω denote the set of equivalent classes of finite dimensional representations of X. If $\delta_1, \delta_2 \in \Omega$ we shall write $\delta_1 \prec \delta_2$ if a member of δ_1 is equivalent to a subrepresentation of a member of δ_2. Let $\mathfrak{r}$ denote the representative ring of X, i.e. the ring of representative functions on X. A function in $\mathfrak{r}$ associated with a representation of class δ will be called a δ-representative function. We denote by $\mathbf{1}$ the class of the trivial representation $x \rightarrow 1$ of X. If δ^* is the class contragredient to $\delta \in \Omega$ it is well known that if $\delta_1, \delta_2 \in \Omega$ and if δ_1 is irreducible then $\delta_1 \prec \delta_2$ if and only if $\mathbf{1} \prec \delta_1^* \otimes \delta_2$. (Here $\otimes$ is the tensor product.) Finally if $\delta \in \Omega$ and $n \geqq 0$ is an integer, we shall denote by δ^n the class of the n-fold tensor product $T \otimes T \otimes \cdots \otimes T$ where $T \in \delta$. By convention, $\delta^0 = \mathbf{1}$.

We begin with the following lemma which must be known.

LEMMA 1. *Let X be a compact connected semisimple Lie group and let $\delta \in \Omega$. Then there exists an integer $n > 0$ such that $\mathbf{1} \prec \delta^n$.*

PROOF. Let T be a member of δ. Set $n =$ degree of T. Then $n > 0$. Letting $H(T)$ stand for the representation space of T, consider the n-fold tensor product $T^n = T \otimes T \otimes \cdots \otimes T$ acting on $H(T) \otimes H(T) \otimes \cdots \otimes H(T)$, let $u_1, u_2, \cdots, u_n$ be a basis of $H(T)$, and let ξ be the vector

$$\xi = \sum_{\sigma} \epsilon(\sigma) u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)}$$

where σ runs over permutations of $1, 2, \cdots, n$ and $\epsilon(\sigma)$ is the signature of σ. Then clearly $\xi \neq 0$. We have

$$T^n(x)\xi = \sum_\sigma \epsilon(\sigma) T(x)u_{\sigma(1)} \otimes T(x)u_{\sigma(2)} \otimes \cdots \otimes T(x)u_{\sigma(n)}.$$

Now let $T(x)u_i = \sum_j T_{ji}(x)u_j$. Then

$$T^n(x)\xi$$
$$= \sum_\sigma \sum_{1\leq j_1,\cdots,j_n\leq n} \epsilon(\sigma) T_{j_1\sigma(1)}(x) \cdots T_{j_n\sigma(n)}(x) \cdot u_{j_1} \otimes u_{j_2} \otimes \cdots \otimes u_{j_n}.$$

In the above sum, the terms which have two of the $j_1, j_2, \cdots, j_n$ equal to one another will vanish when summed over σ because $\epsilon(\sigma)$ is alternating. So only those terms for which $j_1, j_2, \cdots, j_n$ are all different will survive. Then $j_1, j_2, \cdots, j_n$ is a permutation π of $1, 2, \cdots, n$. So

$$T^n(x)\xi$$
$$= \sum_\pi \sum_\sigma \epsilon(\sigma) T_{\pi(1)\sigma(1)}(x) \cdots T_{\pi(n)\sigma(n)}(x) \cdot u_{\pi(1)} \otimes u_{\pi(2)} \otimes \cdots \otimes u_{\pi(n)}$$
$$= \sum_\pi \left(\sum_\sigma \epsilon(\sigma) T_{\pi(1)\sigma(1)}(x) \cdots T_{\pi(n)\sigma(n)}(x) \right) u_{\pi(1)} \otimes \cdots \otimes u_{\pi(n)}$$
$$= \sum_\pi \epsilon(\pi) \det T(x) \cdot u_{\pi(1)} \otimes \cdots \otimes u_{\pi(n)}$$
$$= (\det T(x)) \cdot \xi.$$

Since X is semisimple, $\det T(x) = 1$ for all $x \in X$. Hence $T^n(x)\xi = \xi$ for all $x \in X$. But this means that $1 \prec \delta^n$. Q.E.D.

REMARK. More generally the lemma holds for any unimodular representation of a compact group.

We can now prove the following theorem.

THEOREM 1. *Let X be a compact connected semisimple Lie group. Let G be the group $X \times X$ acting on X by $(u, v): x \to uxv^{-1}$ $u, v, x, \in X$. Let A be a uniformly closed G-invariant subalgebra $\neq \{0\}$ of $C(X)$. Then A is self adjoint and contains the constants.*

PROOF. Since A is stable under the (left) translations by elements of X, and since X is compact, A must be the uniform closure of $\mathfrak{r} \cap A$. To show that A is self-adjoint it is therefore enough to show that $\mathfrak{r} \cap A$ is self-adjoint. Since every function $f \in \mathfrak{r} \cap A$ is a linear combination of representative functions in A associated with finitely many irreducible classes $\delta_1, \delta_2, \cdots, \delta_k \in \Omega$, it is enough to show that if $\delta \in \Omega$ is irreducible and f is a δ-representative function in A then $\bar{f}$ is in A. Let f be such a function. We may obviously assume that $\delta \neq 1$. Since X is semisimple this means that $n =$ degree $\delta > 1$. Now

$\bar{f}$ is δ^*-representative. Lemma 1 says that $1 \prec \delta^n$, so that $\delta^* \prec \delta^{n-1}$. Now since $f \in A$ and A is stable under *both* left and right translations, and since δ is irreducible, *every* δ-representative function is in A. Since A is an algebra and δ^k-representative functions are polynomials in the δ-representative functions, it follows that every δ^k-representative function is in A for each $k > 0$. But $\delta^* \prec \delta^{n-1}$ means that every δ^*-representative function is also δ^{n-1}-representative. It follows that $\bar{f} \in A$ so A is self-adjoint. Since the constants are $\delta \otimes \delta^*$-representative for any $\delta \in \Omega$ it is now easy to conclude that the constants are in A. Q.E.D.

We can now state the following,

THEOREM 2. *Let X be a compact connected semisimple Lie group regarded as a Riemannian symmetric space in the canonical two-sided invariant metric. Let G be the component of the identity in the group $\tilde{G}$ of all isometries of X. Let Δ be the centraliser of G in $\tilde{G}$. Let A be a subalgebra of $C(X)$ which is $\neq \{0\}$, G-invariant, and uniformly closed. Then there exists a subgroup Γ of Δ such that $A = \{f : f \circ \gamma = f$ for all $\gamma \in \Gamma\}$. Γ is unique.*

Conversely every subgroup Γ of Δ gives rise by the above recipe to a G-invariant closed subalgebra of $C(X)$.

The proof is omitted, being very similar to that of Theorem 7.1 of [2].

Added in proof. After this note was written, the author learned from J. Wolf that he has independently arrived at the same results. His proof, along with some other results, will appear in the Pacific Journal of Mathematics.

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