LUSTERNIK-SCHNIRELMAN CATEGORY AND NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

BY FELIX E. BROWDER[1]

Communicated March 8, 1965

Introduction. Let Ω be a bounded, smoothly bounded open subset of R^n (or of a differentiable manifold), f and g two real-valued functionals of the form

$$f(u) = \int F(x, u, \cdots, D^{m-1}u)\, dx, \tag{1}$$

$$g(u) = \int G(x, u, \cdots, D^{m}u)\, dx, \tag{2}$$

defined for r-vector functions u on Ω. Let A and B be the Euler-Lagrange systems for f and g respectively, i.e.

$$Au = \sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} D^\alpha F_{p_\alpha}(x, u, \cdots, D^{m-1}u), \tag{3}$$

$$Bu = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha G_{p_\alpha}(x, u, \cdots, D^{m}u). \tag{4}$$

In a preceding note [**3**], we observed that under assumptions of polynomial growth on F, G, F_{p_α}, and G_{p_α} in u and its derivatives, ellipticity and positivity for B, and positivity for A, there exists an eigenfunction of the pair (A, B), i.e. a solution u of the equation $Bu = \lambda Au$ with λ in R^1, with $f(u)$ prescribed and u satisfying a null variational boundary condition corresponding to a given closed subspace V of a Sobolev space $W^{m,p}(\Omega)$.

It is our object in the present note to summarize the principal results of the writer's paper [**5**], where it is shown that if in addition f and g are even functionals of u, then there exist an infinite number of distinct eigenfunctions u_j with $g(u_j) = c$, prescribed. This result is based in turn upon the estimation from below of the number of critical points of a real-valued function f on an infinite dimensional Finsler manifold M in terms of the Lusternik-Schnirelman category of M.

1. Let M be an infinite-dimensional manifold of class C^2 modelled on the reflexive Banach space B (cf. [**7**], $T(M)$ and $T^*(M)$ the tangent and cotangent spaces of M, respectively.

A Finsler structure on M is a function p: $T^*(M) \to R^1$ such that p

---

[1] The preparation of this paper was partially supported by N.S.F. Grant GP 3552.

is an uniformly convex norm on each cotangent space T_x^*, which is equivalent to the B^* norm. Let p_0 be the dual norm function on $T(M)$. Then p is said to be smooth if there exists a mapping j of $T(M)$ into $T^*(M)$ for which

$$\langle u, j(u)\rangle = p_0(u)p(j(u)), \qquad p(j(u)) = p_0(u),$$

with j locally Lipschitzian on the complement of the zero section in $T^*(M)$. A Finsler structure defines a metric on M by letting

$$d(x, y) = \inf_C \int_0^1 p_0(x'(t))\, dt,$$

where C runs through curves $C: I \to M$ with $C(0) = x$, $C(1) = y$. M is said to be complete with respect to p if it is complete in the induced metric.

DEFINITION 1. *Let f be a real-valued C^1 function on M, f' the corresponding section of $T^*(M)$. Then f is said to satisfy condition* (C) *if on each closed subset N of M on which $|f|$ is bounded with N containing no critical points of f, we have $p(f'(x)) \geq d_0 > 0$ for some constant d_0 and all x in N.*

Condition (C) was applied to the study of the Morse theory on infinite-dimensional Riemannian manifolds by Palais and Smale [11], [12], [14] and to Lusternik-Schnirelman category on Riemannian manifolds by J. T. Schwartz [13]. In [5], we connect up condition (C) with ellipticity or monotonicity conditions applied by the writer to nonlinear elliptic boundary value problems in [1], [2], [4].

DEFINITION 2. *Let*

$$\operatorname{cat}_k(M) = \sup_K \{\operatorname{cat}(K, M) \mid K \text{ a compact subset of } M\},$$

where $\operatorname{cat}(K, M)$ *is the least number s of closed subsets $\{K_1, \cdots, K_s\}$ of K such that $K = \bigcup_j K_j$ and each K_j is contractible over M.*

THEOREM 1. *Let p be a smooth Finsler structure on the C^2 manifold M with M complete with respect to p. Suppose that f is a real-valued C^2 function on M with f bounded from below on M and satisfying condition* (C). *Then if $n(f)$ is the number of distinct critical points of f on M, we have* $\operatorname{cat}_k(M) \leq n(f)$.

Theorem 1 is a variant of results in Banach spaces obtained by Lusternik and others in the Russian literature (cf. [6], [8], [15]) and for Riemannian manifolds by Schwartz [13].[2]

[2] *Added in proof.* A more general result than Theorem 1 without the assumption of uniform convexity on the modelling space has been given by R. Palais in his Brandeis Lecture Notes of 1964–65.

We apply and specialize Theorem 1 to obtain the following general result on eigenvalues of gradients in Banach spaces.

THEOREM 2. *Let X be an uniformly convex Banach space of infinite dimension with C^2 norm on its unit sphere. Let f and g be two real-valued even functions on X of class C^2 on $X - \{0\}$ with f bounded from below on X and g bounded on some sphere $\{x \mid \|x\| = d_0\}$. Suppose that all of the following conditions hold for $\|x\| \geqq d_0$:*

(a) *For any set N where $g(x)$ is bounded, there exists $c_0 > 0$ such that $\langle g'(x), x\rangle \geqq c_0 \|g'(x)\| \geqq c_0^2$, $x \in N$.*

(b) *For any set N_1 on which f and g are bounded, there exists c_1 such that $|\langle f'(x), x\rangle| \leqq c_1 \langle g'(x), x\rangle$, $x \in N_1$.*

(c) *If N_2 is a set on which f and g are bounded, $f'(N_2)$ is precompact in X.*

(d) *For each $M > 0$, there exists a compact map C_M of X and a continuous strictly increasing real function c_M on R^1 with $c_M(0) = 0$, such that for $g(x) = g(y) = M$, $f(x) \leqq M$, $f(y) \leqq M$, we have*

$$\|g'(x) - g'(y)\| + \|C_M(x) - C_M(y)\| \geqq c_M(\|x - y\|).$$

Then there exists a constant c_2 such that for each c with $c \geqq c_2$, there exists an infinite sequence of distinct x_j in X with $g(x_j) = c$ for which

$$f'(x_j) = \lambda_j g'(x_j), \qquad \lambda_j \in R^1.$$

2. We now let X be a closed subspace of a Sobolev space $W^{m,p}(\Omega)$ with $p \geqq 2$, and assume the following bounds on F, G, $F_\alpha = F_{p_\alpha}$, $G_\alpha = G_{p_\alpha}$, $F_{\alpha\beta} = F_{p_\alpha p_\beta}$, $G_{\alpha\beta} = G_{p_\beta p_\beta}$:

$$(5) \quad \begin{aligned}
|F(x, \xi)| &\leqq \Big(1 + \sum_{|\alpha| \leq m-1} |\xi_\alpha|^{q_\alpha}\Big) h(x, \pi(\xi)),\\
|G(x, \xi)| &\leqq \Big(1 + \sum_{|\alpha| \leq m} |\xi_\alpha|^{q_\alpha}\Big) h(x, \pi(\xi)),\\
|F_\alpha(x, \xi)| &\leqq \Big(1 + \sum_{|\beta| \leq m-1} |\xi_\beta|^{q_{\alpha\beta}}\Big) h(x, \pi(\xi)),\\
|G_\alpha(x, \xi)| &\leqq \Big(1 + \sum_{|\beta| \leq m} |\xi_\beta|^{q_{\alpha\beta}}\Big) h(x\pi(\xi)),\\
|F_{\alpha\beta}(x, \xi)| &\leqq \Big(1 + \sum_{|\gamma| \leq m-1} |\xi_\gamma|^{q_{\alpha\beta\gamma}}\Big) h(x, \pi(\xi)),\\
|G_{\alpha\beta}(x, \xi)| &\leqq \Big(1 + \sum_{|\gamma| \leq m} |\xi_\gamma|^{q_{\alpha\beta\gamma}}\Big) h(x, \pi(\xi)),
\end{aligned}$$

with

$$q_\alpha = np(n + p(m - |\alpha|))^{-1},$$
$$q_{\alpha\beta} \leqq q_\beta(q_\alpha - 1)q_\alpha^{-1} \text{ (equality only for } |\alpha| = |\beta| = m),$$
$$q_{\alpha\beta\gamma} \leqq q_\gamma(1 - q_\alpha^{-1} - q_\beta^{-1}),$$
$$\pi(\xi) = \left\{\xi_\alpha \,\middle|\, |\alpha| < m - \frac{n}{p}\right\},$$

h a continuous function of x and π for $x \in \Omega$.

Let

$$a(u, v) = \int \sum_{|\alpha| \leq m-1} \langle F_\alpha(x, u, \cdots, D^{m-1}u), D^\alpha v(x)\rangle\, dx,$$

$$b(u, v) = \int \sum_{|\alpha| \leq m} \langle G_\alpha(x, u, \cdots, D^m u), D^\alpha v\rangle\, dx.$$

Then we have:

THEOREM 3. *Suppose that f is bounded from below, f and g even, and in addition to the above conditions, the following three conditions hold:*

(i) *There exist $d_0 > 0$, $c_0 > 0$, such that*

$$b(u, u) \geqq c_0 \text{ for } \|u\| \geqq d_0, \qquad u \in X.$$

(ii) *There exists a positive continuous function c_1 on R^1 such that if $g(u) \leqq N$, $f(u) \leqq N$, then*

$$|a(u, u)| \geqq c_1(N).$$

(iii) *For each $N > 0$, there exists a continuous strictly increasing function c_N on R^1 with $c_N(0) = 0$ such that*

$$b(u, u - v) - b(v, u - v) \geqq c_N(\|u - v\|)\|u - v\|$$

for all u and v in X for which $\|u\|$, $\|v\|$, $f(u)$, $f(v)$ all are $\leqq N$.

Then for some positive constant c_2 and each $c \geqq c_2$, there exists an infinite sequence of distinct u_j in X with

$$g(u_j) = c$$

such that for all v in X, and certain λ_j in R^1,

$$a(u_j, v) = \lambda_j b(u_j, v).$$

Forms of Theorem 3 with condition (iii) replaced by an ellipticity hypothesis in the stricter sense of being dependent only upon the highest order derivatives are given in detail in [5].

BIBLIOGRAPHY

1. F. E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc. **69** (1963), 862–874.

2. ———, *Nonlinear elliptic boundary value problems*. II, Trans. Amer. Math. Soc. **117** (1965), 530–550.

3. ———, *Variational methods for nonlinear elliptic eigenvalue problems*, Bull. Amer. Math. Soc. **71** (1965), 176–183.

4. ———, *Existence and uniqueness theorems for solutions on nonlinear boundary value problems*, Proc. Sympos. Appl. Math. Vol. 17, Amer. Math. Soc., Providence, R. I., 1965; pp 24–49.

5. ———, *Infinite dimensional manifolds and nonlinear elliptic eigenvalue problems*, Ann. of Math. (to appear).

6. M. A. Krasnoselski, *Topological methods in the theory of nonlinear integral equations*, GITTL, Moscow, 1956; English transl., Macmillan, New York, 1964.

7. S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1963.

8. L. A. Lusternik and L. G. Schnirelman, *Topological methods in variational problems*, Trudy Inst. Mat. Mech. Moscow State Univ. (1930), 1–68.

9. J. Milnor, *Morse theory*, Annals of Mathematics Studies, Princeton, 1963.

10. M. Morse, *The calculus of variations in the large*, Amer. Math. Soc. Colloq. Publ. Vol. 18, Amer. Math. Soc., Providence, R.I., 1934.

11. R. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340.

12. R. Palais and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc. **70** (1964), 165–171.

13. J. T. Schwartz, *Generalizing the Lusternik-Schnirelman theory of critical points*, Comm. Pure Appl. Math. **17** (1964), 307–315.

14. S. Smale, *Morse theory and a nonlinear generalization of the Dirichlet problem*, Ann. of Math. **80** (1964), 382–396.

15. M. M. Vainberg, *Variational methods for the study of nonlinear operators*, GITTL, Moscow, 1956.

UNIVERSITY OF CHICAGO