MODELS OF SPACE-TIME

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1. **Introduction.** In [1] we exhibited electron spin as a nonrelativistic geometric property of (a model of) Euclidean 3-space. We now extend our model to one of space-time. The connections between 2 and 4 component spinors become lucid, while the Dirac equation and its relativistic "invariance" properties undergo a fundamental simplification and clarification.

2. **Abstract space-time.** We need first an axiomatic foundation strong enough to support both our mathematical considerations and their applications to physics.

DEFINITION. *An $n+1$ dimensional space-time ($n \geqq 1$) consists of*

(A) *An $n+1$ dimensional vector space V over the real numbers plus a symmetric bilinear real form $A \cdot B$* (inner product) *such that*:

(1) *There exists a vector A with $A \cdot A < 0$.*

(2) *Any 2-dimensional subspace of V contains a vector A with $A \cdot A > 0$.*

(B) *A set χ of objects $p, q, \cdots$* (points *or* "events") *plus a mapping $(p, q) \to p-q$ of $\chi \times \chi$ into V such that*:

(1) $(p-q)+(q-r)=p-r$.

(2) $p-q=0$ *implies* $p=q$.

(3) *Given any point q and any vector A there exists a point p with $p-q=A$.*

Any V satisfying (A) yields a model of space-time (*vector space-time*) on setting $\chi = V$. The Minkowski model $V=\chi=R_M^{n+1}$ consists of all $n+1$-tuples of real numbers $x=(x_1, \cdots, x_n, x_{n+1})$ with $x \cdot y = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$. (When $n=3$, $x_4=ct$, where t is time and c is the velocity of light.) Every $n+1$ dimensional vector space-time is isomorphic to R_M^{n+1}, but this result is physically misleading. Eventually we set $n=3$, $\chi =$ the physical space-time continuum, and $V=\mathfrak{S}_4$, the *spin model of* (vector) *space-time* we shall construct.

3. **The models $\mathfrak{S}_3$ and W_4.** In [1] we defined the spin model $\mathfrak{S}_3$ of Euclidean 3-space as the vector space of self-adjoint linear transformations of trace 0 in a 2-dimensional unitary space H_2 (spin space) plus the operations $A \cdot B = (1/2)(AB+BA)$ and $A \times B = (1/2i)$

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$\cdot(AB-BA)$. (We identify a scalar c with cI, where I is the identity transformation in H_2.) In general we denote the algebra of linear transformations in a vector space E by $B(E)$. We summarize some results of [1] that we need:

Relative to an arbitrary orthonormal basis ϕ_1, ϕ_2 for H_2 any vector A in $\mathfrak{E}_3$ has the matrix representation

$$A \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli matrices. Then $\mathfrak{E}_3$ is 3-dimensional and

$$A \cdot A = A^2 = x_1^2 + x_2^2 + x_3^2 = -\det A.$$

Let SU(2) denote the group of unitary transformations in H_2 of determinant 1 and SO(3), the group of rotations or orthogonal transformations of determinant 1 in $\mathfrak{E}_3$. Given U in SU(2) set $R_UA = UAU^{-1}$ ($A \in \mathfrak{E}_3$). Then R_U is a linear transformation in $\mathfrak{E}_3$, and the mapping $U \rightarrow R_U$ is a 2-to-1 homomorphism of SU(2) onto SO(3).

The obvious extension of $\mathfrak{E}_3$ is the vector space W_4 consisting of *all* self-adjoint linear transformations in H_2. Then for any A in W_4

$$A \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{pmatrix} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 + x_4$$

and $-\det A = x_1^2 + x_2^2 + x_3^2 - x_4^2 \equiv A \cdot A$. W_4 is then a 3+1-dimensional vector space-time, but the corresponding inner product is hybrid:

$$A \cdot B = \tfrac{1}{2}(AB + BA) - \tfrac{1}{2}(\text{trace } B)A - \tfrac{1}{2}(\text{trace } A)B.$$

One can now extend the covering map above by setting SL(2, C) = the group of linear transformations in H_2 of determinant 1, $\mathfrak{L}_+^\uparrow$ = the homogeneous proper orthochronous Lorentz group; i.e., the linear transformations in W_4 that preserve the inner product, have determinant 1, and don't exchange past and future. Given S in SL(2, C) set $M_SA = SAS^*$ ($A \in W_4$). Then M_S is a linear transformation in W_4, and one has the extended

THEOREM 3.1. *The mapping $S \rightarrow M_S$ is a 2-to-1 homomorphism of* SL(2, C) *onto* $\mathfrak{L}_+^\uparrow$.

This result is essentially known in matrix disguise, but the coordinate-free methods of [1] afford a simpler and more incisive proof than is to be found in the literature.

Although its inner product lacks the Jordan form substituting in $\mathfrak{S}_3$, the model W_4 is appropriate to analysis of the Maxwell equations and the Weyl neutrino, as we shall show in a later paper.

4. **The antiquaternion unit** J. What one wants is an element J in $B(H_2)$ with real square and anticommuting with $\mathfrak{S}_3$. But the only element of $B(H_2)$ that anticommutes with $\mathfrak{S}_3$ is 0. For the same reason *no* nonsingular U in $B(H_2)$ yields the space inversion $P: R_U A = UAU^{-1} = -A$ $(A \in \mathfrak{S}_3)$. We are thus led to the following

PROBLEM. Find all antilinear transformations J in H_2 anticommuting with $\mathfrak{S}_3$, in particular those such that $J^2 = \pm 1$.

In an equivalent guise (commutativity of J with the quaternion algebra $Q = [kU: k \geqq 0,\ U \in \mathrm{SU}(2)]$ (cf. [2])) we obtained in [4] the following

SOLUTION. Given an arbitrary orthonormal basis ϕ_1, ϕ_2 in H_2, identify a vector $x_1\phi_1 + x_2\phi_2$ with the column vector

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then every such J is of the form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = J\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \omega\begin{pmatrix} -\bar{x}_2 \\ \bar{x}_1 \end{pmatrix} = \omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \tag{1}$$

whence $J^2 = -|\omega|^2 \neq 1$ and $J^2 = -1$ iff $|\omega| = 1$—i.e., iff J is antiunitary.

The normalized J, $J^2 = -1$, thus obtained is unique up to a phase factor and may be identified with Wigner's nonrelativistic time-inversion operator for particles of spin $\frac{1}{2}$, but the idea goes back to Möbius: The space inversion operator $R_J A = JAJ^{-1} = -A$ $(A \in \mathfrak{S}_3)$ arising is independent of the scalar $\omega \neq 0$, whence one can regard (1) as an antiprojective transformation in homogeneous coordinates. Set $z = x_1/x_2$, $z' = x_1'/x_2'$ to obtain

$$z' = -\bar{z}^{-1}. \tag{2}$$

Now map onto the Riemann sphere, $z \to \xi$, and note that ξ' is antipodal to ξ.

We can now rewrite the defining properties of $\mathfrak{S}_3$ as follows:

$\mathfrak{S}_3$ consists of all T in $B(H_2)$ such that

$$iT = T^* i, \qquad JT = -T^* J, \tag{3}$$

while the identity $A^*JA = (\det A)^- J$ for A in $B(H_2)$ translates the defining properties of SU(2) into:

SU(2) *consists of all* T *in* $B(H_2)$ *such that*

$$T^*iT = i, \quad T^*JT = J. \tag{4}$$

These formulae are independent of the phase factor for the normalized J. *We now pick a distinguished* J. This amounts to putting a complex orientation on H_2 (cf. [4]).

5. **The spin model $\mathfrak{S}_4$ and the group $\mathcal{G}_+^\uparrow$.** *Now let* E_4 *be* H_2 *considered as a real vector space plus the new inner product*

$$\langle x \mid y \rangle_+ = \Re(\langle x \mid y \rangle). \tag{5}$$

E_4 is a 4-dimensional Euclidean vector space. Linear and antilinear transformations in H_2 are then on the same footing as linear transformations in $\mathfrak{S}_4$, betraying their origin only in commutativity or anticommutativity with the now distinguished linear transformation i. $S = T^*$ in $B(H_2)$ implies $S = T^*$ in $B(E_4)$, while the new and old trace and determinant of a T from $B(H_2)$ are connected as follows:

$$\text{trace}_4\, T = 2\Re(\text{trace}_2\, T), \quad \det_4 T = |\det_2 T|^2. \tag{6}$$

DEFINITION. $\mathfrak{S}_4$ *consists of all linear transformations in* E_4 *satisfying* (3).

Clearly $\mathfrak{S}_4$ is a subspace of $B(E_4)$ containing $\mathfrak{S}_3$ and closed under *.

THEOREM 5.1. $\mathfrak{S}_4$ *consists of all elements of* $B(E_4)$ *of the form*

$$T = A + aJ \quad (A \in \mathfrak{S}_3,\ a \text{ real}).$$

Then $T^2 = A^2 - a^2$ and we can set $T_1 \cdot T_2 = \frac{1}{2}(T_1T_2 + T_2T_1)$ to obtain a 3+1 dimensional model of vector space-time.

Let $K = (1+J)/2^{1/2}$. Then K is orthogonal, $K^2 = J$, and $K^8 = 1$.

THEOREM 5.2. *The mapping* $\tau: A \to KAK$ *is an isomorphism of* W_4 *onto* $\mathfrak{S}_4$ *leaving* $\mathfrak{S}_3$ *pointwise fixed and preserving the inner product.*

Since every T in $B(E_4)$ admits a unique decomposition $T = T_1 + T_2$, where T_1, T_2 are respectively linear and antilinear transformations in H_2, the space-time $\mathfrak{S}_4$ splits naturally into space and time.

DEFINITION. $\mathcal{G}_+^\uparrow$ *consists of all linear transformations* T *in* E_4 *satisfying* (4).

THEOREM 5.3. $\mathcal{G}_+^\uparrow$ *is a group containing* SU(2) *and closed under* *.

If $T \in \mathcal{G}_+^\uparrow$ set $L_T A = TAT^{-1}$ $(A \in \mathfrak{E}_4)$. Then L_T is a linear transformation in $\mathfrak{E}_4$, and

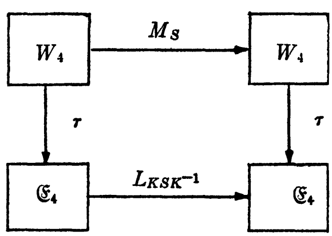
THEOREM 5.4. *The mapping $T \to L_T$ is a 2-to-1 homomorphism of $\mathcal{G}_+^\uparrow$ onto $\mathcal{L}_+^\uparrow$.*

Space-inversion $\boldsymbol{P}$ and time-reversal $\boldsymbol{T}$ arise as follows: $\boldsymbol{P}$: $A \to JAJ^{-1}$, $\boldsymbol{T}$: $A \to iAi^{-1}$. Let $\mathcal{G}$ be the group of linear transformations in E_4 generated by $\mathcal{G}_+^\uparrow$, J, and i.

The connection between 2- and 4-component spinors is then contained in

THEOREM 5.5. *The mapping ν: $S \to KSK^{-1}$ is an isomorphism of* SL(2, *C*) *onto $\mathcal{G}_+^\uparrow$ leaving* SU(2) *pointwise fixed.*

THEOREM 5.6. *The following diagram is commutative:*



Note that $\det_4 KSK^{-1} = \det_4 S = |\det_2 S|^2 = 1$, while $\det J = \det K = 1$ and $\det_4 i = 1$, whence $\mathcal{G}_+^\uparrow$ (or $\mathcal{G}$) and SL(2, C) are subgroups of SL(4, R) whose intersection is SU(2).

$\mathfrak{E}_4$ is also remarkable in that it admits an explicit coordinate-free oriented volume function $\theta(A_1, A_2, A_3, A_4) = \frac{1}{4}\operatorname{trace}_4 (iA_1A_2A_3A_4J)$, reducing to $(1/2i)\operatorname{trace}_2(A_1A_2A_3) = (A_1 \times A_2)\cdot A_3$ when $A_4 = J$ and A_1, A_2, A_3 lie in $\mathfrak{E}_3$ (cf. [3]). Finally, the (Clifford) algebra generated by $\mathfrak{E}_4$ is just $B(E_4)$.

6. **The Dirac operator.** Let $(g_{ij}) = \operatorname{diag}(1, 1, 1, -1)$. Then an ordered orthonormal basis (e) for $\mathfrak{E}_4$ is characterized by the identity

$$e_ie_j + e_je_i = 2g_{ij}. \tag{7}$$

Let E_4^c and $\mathfrak{E}_4^c$ be the respective complexifications of E_4 and $\mathfrak{E}_4$ and consider the expression $\langle Au, v\rangle$, where A runs over $\mathfrak{E}_4$ and u, v run over E_4^c. Since this expression is real linear in A, complex linear in u, and complex antilinear in v, there exists a unique mapping F: $E_4^c \times E_4^c \to \mathfrak{E}_4^c$ such that

$$\langle Au \mid v\rangle = A\cdot F(u, v), \tag{8}$$

and $F(u, v)$ is complex linear in u and complex antilinear in v. In particular, $F(u, Ju)$ lies in $\mathfrak{E}_4$.

Given now any ordered o.n. basis $e_1, \cdots, e_4$ for $\mathfrak{E}_4$ consider smooth functions $\psi: \mathfrak{E}_4 \to E_4^c$ and let

$$(\partial_j\psi)(x) = \lim_{h\to 0} \frac{\psi(x + he_j) - \psi(x)}{h}. \tag{9}$$

Definition. *The Dirac operator* $\mathfrak{D} = e_1\partial_1 + e_2\partial_2 + e_3\partial_3 - e_4\partial_4$. Then $\mathfrak{D}^2 = \partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_4^2$, the d'Alembertian, while the Dirac equation takes the form

$$\mathfrak{D}\psi + \kappa\psi = 0 \qquad (\kappa = mc/\hbar), \tag{10}$$

and the associated charge-current vector $-F(\psi, J\psi)$ satisfies the continuity equation

$$\operatorname{div} F(\psi, J\psi) = 0. \tag{11}$$

Finally the relativistic "invariance" properties of the Dirac equation reduce to simple properties of the Dirac operator $\mathfrak{D}$.

Theorem 6.1 (Passive invariance). $\langle \mathfrak{D}\psi \mid u \rangle = \operatorname{div} F(\psi, u)$ $(u \in E_4^c)$.

If $T \in \mathcal{G}$, let $(\hat{T}\psi)(x) = T\psi(L_{T^{-1}}x) = T\psi(T^{-1}xT)$.

Theorem 6.2 (Active invariance). $\mathfrak{D}\hat{T} = \hat{T}\mathfrak{D}$.

Proofs of the above theorems and some related results will appear elsewhere.

References

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