Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., rev. ed., 1957.
7. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.
8. E. Nelson, *Analytic vectors*, Ann. of Math. 70 (1959), 572–615.
9. L. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, N. J., 1939.
10. I. E. Segal, *A class of operator algebras which are determined by groups*, Duke Math. J. 18 (1951), 221–265.
11. S. Yosida, *On the differentiability and the representation of one parameter semi-group of linear operators*, J. Math. Soc. Japan 1 (1948), 15–21.

PRINCETON UNIVERSITY AND
UNIVERSITY OF CALIFORNIA, BERKELEY

WEAK LEVI CONDITIONS IN SEVERAL COMPLEX VARIABLES[1]

BY AVNER FRIEDMAN

Communicated by R. P. Boas, June 28, 1965

1. **Introduction.** Let $\Omega = \{z;\ z \in \Omega_0,\ \rho(z) < 0\}$ be a bounded domain in C^n, where $\rho \in C^2(\Omega_0)$, Ω_0 a neighborhood of Ω, and let grad $\rho \neq 0$ on $\partial\Omega$. As is well known, if Ω is a domain of holomorphy then for any $x^0 \in \partial\Omega$,

$$(1)\quad L(\rho(x^0), w) \equiv \sum_{j,k=1}^{n} \frac{\partial^2 \rho(x^0)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geqq 0 \quad \text{whenever} \quad \sum_{j=1}^{n} \frac{\partial \rho(x^0)}{\partial z_j} w_j = 0,$$

and, if (1) holds with strict inequality (for $w \neq 0$) then Ω is a domain of holomorphy. (1) is called the *Levi condition* (LC) and, in case of strict inequality, the *strict* LC. One of the consequences of the present work is that the above statement remains true if the assumption $\rho \in C^2$ is replaced by $\rho \in H^{2,\infty}$ (see §2).

In what follows Ω is always given by ρ as above, where $\rho \in C^1(\Omega_0)$, grad $\rho \neq 0$ on $\partial\Omega$.

2. **Definitions.** If ρ has second weak derivatives which belong to $L^p(\Omega_0)$ $(1 < p < \infty)$ then we say that Ω and ρ belong to $H^{2,p}$. Actually we shall only need the derivatives $\partial^2\rho/\partial z_j \partial \bar{z}_k$ to belong to L^p, but then

[1] This work was partially supported by the Alfred P. Sloan Foundation and by Nasa Grant NGR 14-007-021.

necessarily $\partial^2\rho/\partial z_j\partial z_k$ are in L^p in compact subsets.[2] If the weak derivatives $\partial^2\rho/\partial z_j\partial\bar{z}_k$ are continuous, then we say that Ω and ρ belong to $H^{2,\infty}$. The weak derivatives $\partial^2\rho/\partial z_j\partial z_k$, are not, in general, even in L^∞.[3]

Let $t(z)$ be a C^∞ function with support in $|z|<1$, $\int t(z)d\lambda=1$ ($d\lambda$ = Lebesgue measure in C^n) and consider

$$\rho_\epsilon(z) = \frac{1}{\epsilon^{2n}}\int_{\Omega_0}\rho(\zeta)t\left(\frac{z-\zeta}{\epsilon}\right)d\lambda,$$

for $\epsilon>0$ sufficiently small, in some neighborhood Ω_* of $\Omega(\bar{\Omega}_*\subset\Omega_0)$. If $\rho\in H^{2,p}$ ($p<\infty$) then, as $\epsilon\to 0$, $\rho_\epsilon\to\rho$, $D\rho_\epsilon\to D\rho$ uniformly in Ω_*, $\int_{\Omega_*}|D^2\rho_\epsilon-D^2\rho|^p d\lambda\to 0$; if $\rho\in H^{2,\infty}$ then the last relation is replaced by $\partial^2\rho_\epsilon/\partial z_j\partial\bar{z}_k\to\partial^2\rho/\partial z_j\partial\bar{z}_k$ uniformly in Ω_*; D is any first order derivative.

Introduce positive functions $\phi_p(A)$, $\psi_p(A)$ and $\eta(\epsilon)$ such that $\eta(\epsilon)\to 0$ if $\epsilon\to 0$ and

$$\sum_{j,k}\left\{\int_A\left|\frac{\partial^2}{\partial z_j\partial\bar{z}_k}(\rho_\epsilon-\rho)\right|^p d\lambda\right\}^{1/p} \leqq \phi_p(A)\eta(\epsilon), \tag{2}$$

$$\sum_{j,k}\left\{\int_A\left|\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\right|^p d\lambda\right\}^{1/p} \leqq \psi_p(A). \tag{3}$$

If $\rho\in H^{2,p}$ ($1<p\leqq\infty$) then such functions clearly exist. We take

$$\phi_p(A) = \psi_p(A) = C(\text{vol. } A)^{1/p} \text{ if } \rho\in H^{2,\infty}\ (C = \text{const.} \neq 0). \tag{4}$$

3. Levi conditions.

DEFINITION 1. Let $\Omega\in H^{2,p}$. We say that the *weak Levi condition* with index p (WLC$_p$) holds at $x^0\in\partial\Omega$ if for any $\epsilon>0$ there exist δ such that

$$\begin{aligned}&\int_{B_\delta} L(\rho(z), w(z))\,d\lambda \\ &\geqq -\epsilon\{\phi_p(V_w)+\psi_p(V_w)+(\text{vol. } V_w)^{1/p}\}\left\{\int_{B_\delta}|w(z)|^{2q}d\lambda\right\}^{1/q},\end{aligned} \tag{5}$$

for any $w\in C^0(B_\delta)$ satisfying

[2] This follows from the inequality (z 1-dimensional) (*)$\int_A|\partial w/\partial z|^p dz\wedge d\bar{z}\leqq$const. $\int_B(|\partial w/\partial\bar{z}|^p+|w|^p)dz\wedge d\bar{z}$ where A is relatively compact to B. To prove(*) use integral representation for w (which may be assumed to have compact support) in terms of $\partial w/\partial\bar{z}$ and L^p estimates for singular integrals.

[3] Indeed, in the contrary case, an argument involving the closed graph theorem yields an inequality of the form l.u.b.$_A|\partial^2 w/\partial z^2|\leqq$const. l.u.b.$_B(|\partial w/\partial z^2\partial\bar{z}|+|\partial w/\partial z|+|\partial w/\partial\bar{z}|+|w|)$ (the same notation as in footnote 1) which is impossible (by [2]).

$$(6)\qquad \sum_{j=1}^{n} \frac{\partial\rho(z)}{\partial z_j} w_j(z) = 0 \qquad \text{in } B_\delta.$$

Here V_w is the support of w, $1/q+1/p=1$, $B_\delta=B(x^0, \delta)$ is the ball with center x^0 and radius δ, and δ is sufficiently small such that $B_\delta\subset\Omega_*$ and grad $\rho\neq 0$ in B_δ.

If we require w to have weak derivatives in L^p, then all the results below remain unchanged; (5) then takes the equivalent form

$$(7)\qquad \begin{aligned} &\sum_{j,k}\int_{B_\delta} \frac{\partial\rho}{\partial z_j}\frac{\partial w_j}{\partial \bar z_k}\bar w_k\, d\lambda \\ &\leqq \epsilon\{\phi_p(V_w)+\psi_p(V_w)+(\text{vol. } V_w)^{1/p}\}\left\{\int_{B_\delta}|w|^{2q}d\lambda\right\}^{1/q}. \end{aligned}$$

Note however that if ρ is not assumed to have second weak derivatives in L^p then there may not exist any w with first derivatives in L^p which satisfies (6).

Definition 2. Let $\Omega\in H^{2,p}$. We say that the WLC$_p I$ holds at $x^0\in\partial\Omega$ if for any $\epsilon>0$, $\delta>0$, there is a point $\bar x\in\partial\Omega\cap B_\delta$ and $0<\mu<\delta-|x^0-\bar x|$ such that (5) holds with B_δ replaced by $B(\bar x, \mu)$, for any w satisfying (6) in $B(\bar x, \mu)$.

Definition 3. Let $\Omega\in H^{2,p}$. If for some positive ϵ_0, δ and for any set $A\subset B_\delta$

$$(8)\qquad \int_A L(\rho, w)\, d\lambda \geqq \epsilon_0\{\phi_p(A)+\psi_p(A)+(\text{vol. } A)^{1/p}\}\left\{\int_A |w|^{2q}d\lambda\right\}^{1/q}$$

for any $w\in C^0(A)$ satisfying (6) in A and such that $1\leqq|w(z)|\leqq 2$ in A, then we say that the *strict* WLC$_p$ holds. The strict WLC$_p I$ is defined similarly.

Note that if we omit the condition $1\leqq|w|\leqq 2$ and replace A by V_w, where $w\in C^0(B_\delta)$, then we obtain a condition which, although analogous to (5), is too restrictive if $p<\infty$, as seen by applying it to a sequence of w's whose support is B_δ but whose limit has its support in B_η for any given $0<\eta<\delta$. Note also that all the results below remain unchanged if in (8) we replace $(\text{vol. } A)^{1/p}\{\int|w|^{2q}d\lambda\}^{1/q}$ by $\int|w|^2 d\lambda$. Finally, all the results remain true if we modify Definitions 1, 2 and 3 by taking $w(x)\equiv$const. such that (6) holds at one point of B_δ, $B(\bar x, \mu)$ and A respectively.

4. Statement of results.

Theorem 1. *Let* $\Omega\in H^{2,p}$ $(1<p\leqq\infty)$. *If* $\rho\in H^{2,\infty}(N)$ *where* N *is some neighborhood of* x^0, *then* (i) *the* WLC$_p$, WLC$_p I$ *and the* LC *at* x^0

coincide, and (ii) *the strict* WLC_p, *strict* WLC_pI *and the strict* LC *at* x^0 *coincide.*

THEOREM 2. *Let* $\Omega\in H^{2,p}$ $(1<p\leqq\infty)$. *If* $\partial\Omega$ *satisfies the strict* WLC_p *then* Ω *is a domain of holomorphy.*

THEOREM 3. *Let* $\Omega\in H^{2,p}$ $(1<p\leqq\infty)$. *If* Ω *is a domain of holomorphy then* $\partial\Omega$ *satisfies the* WLC_pI.

Set $K=\bar{\Omega}$. Denote by $A(K)$ the algebra of uniform limits of functions holomorphic on K, by $S(K)$ the space of its maximal ideals, and by $\Gamma(K)$ its Shilov boundary. If K is S_δ, i.e., if $K=\bigcap_{m=1}^{\infty}\Omega_m$ where Ω_m are domains of holomorphy then (Rossi [4]) $S(K)=K$ and, if $\partial\Omega\in C^2$, $\Gamma(K)=\mathrm{Cl}(P(K))$ where $P(K)$ is the set of points of $\partial\Omega$ where the strict LC holds and "Cl" means "closure of." Furthermore, for each $\bar{x}\in P(K)$ there exists f holomorphic on K with

$$|f(\bar{x})| > |f(z)| \qquad \text{for all } z\in K,\ z\neq\bar{x}. \tag{9}$$

THEOREM 4. *If* K *is* S_δ, $\Omega\in H^{2,p}$ *and* $\partial\Omega$ *satisfies the strict* WLC_p, *then* $\Gamma(K)=\partial\Omega$.

Theorem 4 can be extended to the case where the strict WLC_p holds only on a part (say E) of $\partial\Omega$, provided $\rho\in C^2$ in a neighborhood of $\partial\Omega\backslash\bar{E}$.

Finally, Theorems 2–4 extend to domains Ω which are only locally given by functions ρ in $H^{2,p}$. One uses a partition of unity in conjunction with the arguments given below.

5. **Proofs outline.** To prove $\mathrm{LC}\Rightarrow\mathrm{WLC}_p$, write

$$L(\rho(z),w(z)) = L(\rho(z)-\rho(x^0),w(z)) + L(\rho(x^0),w(z)-w^0) + L(\rho(x^0),w^0)$$

and, given $\eta>0$, determine δ, w^0 such that $|w(z)-w^0|\leqq\eta|w(z)|$ in B_δ, and make use of (4) and of Hölder's inequality. To prove $\mathrm{WLC}_pI\Rightarrow\mathrm{LC}$, assume $L(\rho(x^0,)\ w^0)<-\epsilon_0|w^0|^2$ and derive $L(\rho(z),w(z))<-\epsilon_0|w^0|^2/2$ (w, w^0 related as above), then integrate. Since $\mathrm{WLC}_p\Rightarrow\mathrm{WLC}_pI$, (i) follows. The proof of (ii) is similar.

To prove Theorem 2, take $\Omega_m=\{z;\ z\in\Omega_*,\ \rho_m(z)\equiv\rho_{\epsilon_m}(z)-\delta_m<0\}$ with $\epsilon_m\to 0$, $\delta_m\to 0$, such that, $\Omega_m\supset\Omega_{m+1}$, and let $x^0\in\partial\Omega$ and ϵ_0, δ as in Definition 3. Let $m\geqq m_0$ be such that $B_\delta\cap\partial\Omega_m\neq\emptyset$ and take $A\subset B_\delta$, $A\cap\partial\Omega_m\neq\emptyset$. Given $W\in C^0(A)$ satisfying (6) with ρ replaced by ρ_m, choose $w\in C^0(A)$ satisfying (6), with $|W(z)-w(z)|\leqq\mu_m|W(z)|$, $\mu_m\to 0$ if $m\to\infty$. Writing

$$\int_A L(\rho_m, W)\, d\lambda = \int_A L(\rho, w)\, d\lambda + \int_A L(\rho_m - \rho, W)\, d\lambda + \int_A L(\rho, W - w)\, d\lambda$$

and using Hölder's inequality and (8), derive (8) with ρ, w, ϵ_0 replaced by ρ_m, W, $\epsilon_0/2$ provided $m \geqq m_1$. Thus, the strict WLC_p (and, by Theorem 1, the LC) holds at each point $\bar{x} \in \partial\Omega_m \cap B_\delta$. By the Heine-Borel Theorem, the LC then holds at all the points of $\partial\Omega_m$, for all large m; thus these Ω_m (and, necessarily, also Ω) are domains of holomorphy.

To prove Theorem 3, let $x^0 \in \partial\Omega$ and ϵ, δ as in Definition 2. Choose $\phi \in C^\infty$ ($\mathbf{C}^n$) satisfying: $\phi = 1$ in $B(x^0, \delta/2)$, $\phi = 0$ outside $B(x^0, \delta)$, $0 \leqq \phi \leqq 1$, elsewhere. Take $\tilde{\rho}_\gamma = \rho_\gamma - A_\gamma + B_\gamma \phi(z)$ with $A_\gamma \to 0$, $B_\gamma \to 0$ if $\gamma \to 0$, $B_\gamma > 0$, such that the domain $\tilde{\Omega}$, defined by $\tilde{\rho}_\gamma < 0$, is contained in Ω and $\partial\tilde{\Omega}$ touches $\partial\Omega$ at some point $\bar{x} \in B(x^0, \delta)$. Observing that not all the functions holomorphic in $\tilde{\Omega}$ can be continued into a neighborhood of $\bar{x}$, a result of Lewy [3] (see also [1, Chapter 2]) implies

$$L(\tilde{\rho}_\gamma(\bar{x}), w) \geqq 0 \quad \text{whenever} \quad \sum \frac{\partial \tilde{\rho}_\gamma(\bar{x})}{\partial z_j} w_j = 0.$$

Now use arguments as in the previous two proofs.

To prove Theorem 4, it suffices to show that for any $x^0 \in \partial\Omega$ and $\delta > 0$ there exists a point $\bar{x} \in \partial\Omega \cap B_\delta$ for which (9) holds. With ϕ as before, construct $\hat{\rho}_\gamma = \rho_\gamma - A'_\gamma - B'_\gamma \phi$ ($A'_\gamma \to 0$, $B'_\gamma \to 0$, $B'_\gamma > 0$) such that $\hat{\Omega}$, defined by $\hat{\rho}_\gamma < 0$, contains Ω and $\partial\hat{\Omega}$ touches $\partial\Omega$ at some point $\bar{x}$ in $\partial\Omega \cap B_\delta$. Show that the result (9) of Rossi can be applied in $\hat{\Omega}$.

References

1. L. Hörmander, *Lectures on functions of several complex variables*, Van Nostrand, Princeton, N. J., 1965 (to appear).

2. K. de Leeuw and H. Mirkil, *Majorations dans L_∞ des opérateurs différentiels à coefficients constants*, C. R. Acad. Sci. Paris **254** (1962), 2286–2288.

3. H. Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular function of two complex variables*, Ann. of Math. **64** (1956), 514–522.

4. H. Rossi, *Holomorphically convex sets in several complex variables*, Ann. of Math. **74** (1961), 470–493.

Northwestern University