A GENERALIZATION OF A THEOREM OF NEHARI

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In 1956 Nehari, [N.1] showed that the singular points t_0 of the Legendre series,

$$(1)\qquad \phi(t) = \sum_{n=0}^{\infty} a_n P_n(t), \quad |t+1| + |t-1| < \rho + \frac{1}{\rho},$$

where $\lim \sup_{n\to\infty} |a_n|^{1/n} = \rho^{-1} < 1$, are related to the singular points ζ_0 of the associated power series,

$$(2)\qquad f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \qquad |\zeta| < \rho^{-1},$$

by the formula $t_0 = \frac{1}{2}(\zeta_0 + 1/\zeta_0)$, providing $t_0 \neq \pm 1$. The purpose of this note is to announce similar results concerning the singularities of functions $\phi(z)$ defined by series of the form $\sum_{n=0}^{\infty} a_n v_n(z)$, where the $v_n(z)$ are normalized eigenfunctions of the Sturm-Liouville system

$$(3)\qquad \begin{aligned} v''(z) + (\rho^2 - q(z))v(z) &= 0, \\ v'(0) + hv(0) = v'(\pi) + Hv(\pi) &= 0. \end{aligned}$$

Indeed we are able to establish the following result.

THEOREM. *Let $q(z) \in C^\infty[0, \pi]$, let the $v_n(z)$ be the set of normalized eigenfunctions of the Sturm-Liouville system* (3), *and let $\{a_n\}$ be a sequence of complex numbers such that* $\lim \sup_{n\to\infty} |a_n|^{1/n} = \rho^{-1} < 1$. *Furthermore, let us introduce the pair of analytic function elements defined in a neighborhood of the origin,*

$$(A)\qquad f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \qquad |\zeta| < \rho^{-1};$$

$$(B)\qquad \psi(t) = \sum_{n=0}^{\infty} a_n u_n(t), \qquad |t+1| + |t-1| < \rho + \frac{1}{\rho},$$

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where $u_n(\cos z) \equiv v_n(z)$. *Then, providing* $t \neq \pm 1$, *the function* $\psi(t)$, *defined above is singular at those points* $t = \frac{1}{2}(\alpha + 1/\alpha)$, *where* $\zeta = \alpha$ *is a singular point of the series* $f(\zeta)$.

SKETCH OF PROOF. The method of proof parallels that given originally by Nehari [**N.1**] which has been extended by Gilbert [**G.1–4**], and Gilbert and Howard [**G.H.1–2**], who have used these ideas in conjunction with Bergman's integral operator method [**B.1–2**] for the study of partial differential equations. (See also the survey paper by Gilbert, Howard and Aks [**G.H.A.1**].)

We are able to introduce an integral operator $\mathfrak{F}[f]$, which maps power series (A) onto the eigenfunction series (B). Furthermore, we also are able to construct an inverse operator $\mathfrak{F}^{-1}[\psi]$, which maps $\psi(t)$ onto $f(\zeta)$. These operators may be seen to have the form given below,

$$\mathfrak{F}[f] \equiv \int_{|\zeta|=\rho_0} K(t, \zeta) f(\zeta) \frac{d\zeta}{\zeta},$$

where $1 < \rho_0 < \rho$, and

$$\mathfrak{F}^{-1}[\psi] \equiv \int_{-1}^{+1} K(t, \zeta^{-1}) \psi(t)\, dt,$$

where the integration is along the real axis, and where $K(t, \zeta) \equiv \sum_{n=0}^{\infty} u_n(t) \zeta^{-n}$. By the use of the elementary Hartogs' theorem [**B.M.1**], and known appraisals for the $v_n(z)$ (and hence for the $u_n(t)$) as $n \to \infty$ we are able to establish that $K(t, \zeta^{-1})$ is a holomorphic function of two complex variables in certain product domains; consequently, we may consider the above integrals as Cauchy integrals.

We are also able to determine the first analytic set (moving outwards from the origin in the ζ-plane) on which $K(t, \zeta^{-1})$ is singular, as given by $\{\zeta^2 - 2\zeta t + 1 = 0\}$. Using this information plus the argument used by Hadamard [**H.1**], [**N.1**] in his proof of the "multiplication of singularities" theorem, allows us to establish the fact that if $f(\zeta)$ is singular at $\zeta = \alpha$, ($|\alpha| = 1/\rho$), then in the compact ellipse, $|t+1| + |t-1| \leqq \rho + 1/\rho$, $\psi(t)$ is regular for all points $t \neq \frac{1}{2}(\alpha + 1/\alpha)$. Correspondingly, we are able to show if $\psi(t)$ has a singularity on the boundary of the above ellipse, say at $t = \sigma$, ($\sigma = \frac{1}{2}(\alpha + 1/\alpha)$), then $f(\zeta)$ is regular at all points $\zeta \neq \alpha$, such that $|\zeta| \leqq |\alpha|$. Combining these results yields the above theorem.

REMARK. We have also been able to find analogous results for eigenfunction expansions associated with quite general nth order Sturm-Liouville systems. These results along with the details of the above proof will be published elsewhere.

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