SUBSPACES OF $C(S)_\beta$, THE SPACE (l^∞, β), AND (H^∞, β)

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The author has shown [2] that if S is a paracompact locally compact space then every β-weak * countably compact subset of $M(S)$ is β-equicontinuous (see [2] for definitions and notation). If we define a *strong Mackey space* to be a topological vector space E such that every weak * compact (not necessarily convex and circled) subset of E^* is equicontinuous, then $C(S)$ with the strict topology β is a strong Mackey space whenever S is paracompact. A natural problem is to characterize those subspaces of $C(S)$ which are (strong) Mackey spaces if they have the relative strict topology and if $C(S)_\beta$ is a (strong) Mackey space. In particular, we may ask this question for a paracompact space S.

Along these lines it is known that the completion of a Mackey space is a Mackey space, but the converse is false. In fact, $C(S)_\beta$ is the completion of $C_0(S)_\beta$, but $C_0(S)_\beta$ is never a Mackey space (unless S is compact), since the norm topology is always stronger than β and yields the same adjoint $M(S)$.

At present we have no solution to our question, but we can give an answer in the case where S is the space of positive integers. Also, we show that H^∞, the space of bounded analytic functions on the open unit disk D, is not a Mackey space if it has the relative β topology, even though $C(D)_\beta$ is a strong Mackey space.

The difficulties encountered in attacking the general problem may be visualized as follows. Let E be a subspace of $C(S)$ and $i: E_\beta \to C(S)_\beta$ the injection map, with $i^*: M(S) \to E_\beta^*$ its adjoint. In order to show that a subset $H \subset E_\beta^*$ is β-equicontinuous it is necessary and sufficient to show that there is a β-equicontinuous subset $H_1 \subset M(S)$ such that $i^* H_1 = H$. Therefore, if $C(S)_\beta$ is a Mackey space and $H \subset E_\beta^*$ is β-weak * compact convex and circled, then to show that H is β-equicontinuous we must find a β-weak * compact convex circled set $H_1 \subset M(S)$ such that $i^* H_1 = H$. Since E_β^* with its β-weak* topology is topologically isomorphic to a quotient space of $M(S)$, it would seem that what is needed is a version of a theorem of Bartle and Graves [4, p. 375] where both domain and range have their β-weak* topolo-

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gies. Unfortunately, no such theorem is available in general, although in the special case of l^∞ one can use this theorem to great advantage.

In [2] we proved that (l^∞, β) is a strong Mackey space by using the fact that in l^1 weak and strong convergence of sequences are equivalent. It is not hard to prove this latter result if we assume that (l^∞, β) is a strong Mackey space. This will be exactly what is needed to characterize those subspaces of (l^∞, β) which are Mackey spaces.

LEMMA 1. *Let E be a β-closed subspace of l^∞. Then E_β is semi-reflexive and E_β^* with its strong topology (=the topology of uniform convergence on bounded subsets of E_β) is a Banach space. Hence, the β-weak * topology on E_β^* is exactly the weak topology which it has as a Banach space.*

THEOREM 2. *A subset of E_β^* is β-equicontinuous if and only if it is norm conditionally compact (its norm closure is norm compact).*

THEOREM 3. *If E is a β-closed subspace of l^∞ then the following are equivalent*: (a) *E_β is a Mackey space*; (b) *E_β is a strong Mackey space*; (c) *every β-weak * compact set in E_β^* is norm compact*; (d) *every β-weak * convergent sequence in E_β^* is norm convergent.*

Now let us turn our attention to H^∞. Theorems 4 and 5 below are due to Shields and Rubel [5], but we present them here because we obtain them by a different method and because they form a direct path to our result that (H^∞, β) is not a Mackey space.

We follow a method of Brown, Shields, and Zeller [1], and obtain a sequence $\{a_n\}_{n=1}^\infty$ in D, having no limit points in D, such that for $f \in H^\infty$, $\|f\|_\infty = \sup\{|f(a_n)| : n \geqq 1\}$. If E is the subspace of l^∞ consisting of all sequences $\{f(a_n)\}_{n=1}^\infty$ where $f \in H^\infty$, and we define $T: H^\infty \to E$ by $T(f) = \{f(a_n)\}$, then T is an isometry. Since $\{a_n\}$ has no limit points $T: (H^\infty, \beta) \to E_\beta$ is continuous, a fact which is crucial in our development.

THEOREM 4. *A subset of H^∞ is β-compact if and only if it is β-closed and bounded.*

THEOREM 5. *If I is a linear functional on H^∞ which is β-continuous on the unit ball then it is β-continuous on H^∞. Hence $(H^\infty, \beta)^*$ with its strong topology is a Banach space and H^∞ is its adjoint.*

PROOF. Consider the map T defined above; since T is continuous and ball H^∞ is β-compact we have that ball $E = T(\text{ball } H^\infty)$ is β-compact in l^∞. Thus E is $\sigma(l^\infty, l^1)$ (the weak topology on l^∞ induced by l^1) closed (see [3, p. 429]); hence E is β-closed in l^∞. Also $I \circ T^{-1}$ is β-continuous on ball E and so $I \circ T^{-1} \in E_\beta^*$. But then $I = T^*(I \circ T^{-1}) \in (H^\infty, \beta)^*$, and the proof is easily completed.

COROLLARY 6. *A linear functional I on H^∞ is β-continuous if and only if there is a Lebesgue integrable function g on $[-\pi, \pi]$ such that $I(f) = (1/2\pi)\int_{-\pi}^{\pi} f(e^{i\theta})g(\theta)d\theta$ for all $f \in H^\infty$.*

THEOREM 7. *A subset of $(H^\infty, \beta)^*$ is β-equicontinuous if and only if it is norm conditionally compact.*

COROLLARY 8. *If $\{I_n\}$, I are in $(H^\infty, \beta)^*$ then $I_n \to I$ in norm if and only if* (1) *$I_n \to I$ (β-weak *) and* (2) *$\{I_n\}$ is β-equicontinuous.*

THEOREM 9. *(H^∞, β) is not a Mackey space.*

PROOF. It is enough to show that (H^∞, β) is not a strong Mackey space. For each $n \geqq 1$ let $I_n(f) = (1/2\pi)\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta = \hat{f}(n)$ for all f in H^∞. Then $I_n \in (H^\infty, \beta)^*$ and $I_n \to 0$ (β-weak *). But $\|I_n\| = 1$ for all $n \geqq 1$ and hence, by the preceding corollary, $\{I_n\}$ is not β-equicontinuous. This concludes the proof.

It is not difficult to show that if N is a norm closed subspace of l^1 such that there is a bounded projection of l^1 onto N, then E_β is a Mackey space where $E = N^\perp \subset l^\infty$ (it is not known if the converse is true). Now if E is the image of H^∞ under the map T described prior to Theorem 4, it is easy to see that E_β is not a Mackey space by using Theorem 9. Thus, if $N = E^\perp \subset l^1$ then no bounded projection of l^1 onto N exists.

Finally let us close by saying that in addition to the uses made of the strict topology we find it interesting in itself. There are no examples known to the author of a topological vector space, with an intrinsically defined topology, which is a Mackey space except by virtue of some other formally stronger property (e.g. barrelled, metric, etc.). However it is easy to see that $C(S)_\beta$ is barrelled, bornological or metric if and only if S is compact. Also we have an example of a semi-reflexive Mackey space (l^∞, β) which has a closed subspace which is not a Mackey space (the image E of H^∞ under the mapping discussed above).

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