NONLINEAR ELLIPTIC FUNCTIONAL EQUATIONS IN NONREFLEXIVE BANACH SPACES

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Consider a strongly elliptic quasilinear system of partial differential operators on an open subset Ω of R^n in generalized divergence form:

$$(1) \qquad A(u) = \sum_{|\alpha| \leq m} D^\alpha A_\alpha(x, u, \cdots, D^m u).$$

If the coefficient functions A_α have at most polynomial growth in u and its derivatives, we have shown in [1] how existence theorems for solutions of variational boundary value problems for the equation $A(u) = f$ may be obtained from monotonicity assumptions on the nonlinear Dirichlet form of A, with extensions to weaker ellipticity assumptions in [3], [4], [9], and [15]. These results extend earlier theorems of Višik [18] obtained by other arguments, but (despite an apparent remark to the contrary in [15]) do not extend automatically to the case of rapidly increasing coefficients treated by Višik in [19]. The crucial point for this more general case is that the Banach spaces in which the problems are appropriately formulated are derived from nonreflexive Orlicz spaces [14] and are themselves nonreflexive and nonseparable. On the other hand, the basic treatments of monotone and semimonotone operator equations have been carried out in reflexive Banach spaces (and in [15], it is even essential to consider only separable spaces).

It is our object in the present note to outline a treatment of these more general problems in nonreflexive spaces, using the remark already made by the writer in the final section of [4] on the useful properties of the weak* topology on a conjugate space. We formulate this treatment in a very general context of functional rather than simply differential equations, and apply to the resulting operator equations a new theorem on perturbation of semimonotone operators (Theorem 2 below) which is particularly efficacious in this general context. This result is extended and applied in more detail in [11] to include all the elliptic results of [1], [3], [4], [9], [15], [18], and [19], and a corresponding extension to unbounded operators and to monotone operators on convex sets gives a significant extension of

the results on nonlinear initial value problems developed in [2], [5], [7], and [9], (as well as of recent results for monotone operators on convex sets of the writer [10] and Hartman and Stampacchia [13]).

1. Suppose that for each multi-index α with $|\alpha| \leq m$, we are given a Banach space X_α of distributions on Ω with its dual space X_α^* also a space of distributions and $C_c^\infty(\Omega) \subset X_\alpha^*$. We set $X = \prod_\alpha X_\alpha$, $X^* = \prod_\alpha X_\alpha^*$. Corresponding to this family of spaces, we have the generalization of the Sobolev spaces given by

$$W_{X^*} = \{u \mid u \in D'(\Omega), D^\alpha u \in X_\alpha^* \text{ for } |\alpha| \leq m\}.$$

Let ζ be the mapping of W_{X^*} into X^* given by $\zeta(u) = \{D^\alpha u \mid |\alpha| \leq m\}$. We norm W_{X^*} so that ζ is an isometry. Then ζ identifies W_{X^*} with a weak* closed subspace of X^*.

The data of a nonlinear elliptic functional equation are the following: We are given a subspace V_0 of W_{X^*} such that $\zeta(V_0)$ is weak* closed in X^*, an element f of X, and a continuous mapping A_α for each α of X^* into X_α. We define the nonlinear Dirichlet form $a(u, v)$ for each u and v in V_0 by

$$(2) \qquad\qquad a(u, v) = (A(\zeta(u)), \zeta(v))$$

where A is the product map of X^* into X with components A_α and $(\cdot, \cdot)$ denotes the pairing between X and X^*. The corresponding boundary problem asks for an element u in V_0 such that for all v in V_0, we have:

$$(3) \qquad\qquad a(u, v) = (f, \zeta(v)).$$

THEOREM 1. *Suppose that the Dirichlet form $a(u, v)$ satisfies the two conditions*:

(1) *For all u, v of V_0*

$$a(u, u - v) - a(v, u - v) \geq 0.$$

(2) *As $\|u\| \to +\infty$,*

$$a(u, u)/\|u\| \to +\infty.$$

Then the above boundary value problem has a solution for every f in X.

PROOF OF THEOREM 1. Let V_0^0 be the polar of $\zeta(V_0)$ in X, and let V be the quotient space X/V_0^0, τ the natural map of X on V. Then τ^* is an isometric map of V^* into X^*, and by the weak* closedness of $\zeta(V_0)$, it follows that $\tau^*(V^*) = \zeta(V_0)$. Hence, the mapping $(\tau^*)^{-1}\zeta$ is a well-defined isometric isomorphism of V_0 on V^*, and if for u and v in V_0, we set $u^* = (\tau^*)^{-1}\zeta(u)$, $v^* = (\tau^*)^{-1}\zeta(v)$, we see that

$$a(u, v) = (A(\zeta(u)), \zeta(v)) = (A\tau^*u^*, \tau^*v^*).$$

Hence the condition that u be a solution of the functional equation (3) above is equivalent to the condition that the corresponding u^* satisfy:

(4) $$(A\tau^*u^*, \tau^*v^*) = (f, \tau^*v^*)$$

for all v^* in V^*, i.e.

$$(\tau A\tau^*(u^*), v^*) = (\tau(f), v^*),$$

or $T(u^*) = \tau A\tau^*(u^*) = \tau(f)$, with $T = \tau A\tau^*$.

The conditions (1) and (2) above are equivalent to the following conditions for the continuous mapping T of V^* into V:

(1)′ For each pair f and g of V^*,

$$(Tf - Tg, f - g) \geq 0.$$

(2)′ As $\|f\| \to + \infty$, $(Tf, f)/\|f\| \to + \infty$.

The original problem will have a solution for a given f in X if the corresponding $\tau(f)$ lies in the range of T. However, the range of T is all of V by Theorem 5 of [4]. q.e.d.

2. A corresponding translation process for the case where we distinguish between the dependence of the Dirichlet form $a(u, v)$ on the highest and the lower order derivatives of u reduces the corresponding existence result to Theorem 2 below. We preface the statement of the theorem with the following definition: A Banach space V is said to be *improvable* if there exists a continuous map M of V^* into V mapping bounded sets into bounded sets which is monotone (i.e. $(Mu - Mv, u-v) \geq 0$ for all u and v), coercive (i.e. $(Mu, u)/\|u\| \to + \infty$ as $\|u\| \to + \infty$) and such that if for a sequence $\{u_k\}$ in V^*, we have $(Mu_k - Mu, u_k - u) \to 0$, then u_k converges strongly to u.

THEOREM 2. *Let V be an improvable Banach space, T a mapping of V^* into V such that $(Tu, u)/\|u\| \to + \infty$ as $\|u\| \to + \infty$. Suppose that there exist mappings S of $V^* \times V^*$ into V and C of V^* into V for which $T(u) = S(u, u) + C(u)$ for all u in V^* and which satisfy the following conditions:*

(1) *For each fixed v in V^*, $S(\cdot, v)$ is a continuous monotone mapping of V^* into V, i.e. $(S(u_1, v) - S(u, v), u - u_1) \geq 0$ for all u and u_1 in V^*.*

(2) *For each fixed u in V^*, $S(u, \cdot)$ is completely continuous, i.e. continuous from the weak* topology of V^* to the strong topology of V.*

(3) *C is compact, i.e. C is continuous from V^* to V and maps each bounded subset of V^* into a strongly precompact subset of V.*

(4) *S dominates C in the following sense: If $\{u_k\}$ is a sequence in*

V^* with u_k converging weak* to u, and if $(S(u_k, u_k) - S(u, u_k), u_k - u)$ $\to 0$, then $C(u_k)$ converges weakly to $C(u)$.

Then $R(T)$, the range of T, is all of V.

COROLLARY TO THEOREM 2. *The condition* (4) *may be replaced by the weaker condition*:

(4)$'$ *For every closed ball B about the origin in V^*, $T(B)$ is closed in V.*

REMARK. It can be shown by simple examples that (1), (2), and (3) alone do not imply the conclusion that $R(T) = V$, even when V is a Hilbert space.

PROOF OF THEOREM 2. Since we may subtract a fixed element w of V from $T(u)$ without altering the hypotheses, it suffices to show that $0 \in R(T)$.

We show first that under hypotheses (1), (2), and (3), condition (4) implies (4)$'$. In the subsequent argument, only condition (4)$'$ will be applied. We use the following fact in this and later proofs (cf. [4]): If Q is a continuous map of V^* into V and if for given u_0 in V^*, w_0 in V, and all u in V^*, we have $(Q(u) - w_0, u - u_0) \geqq 0$, then $Q(u_0) = w_0$.

PROOF OF (4)$'$ FROM (4). Let $\{u_k\}$ be a sequence in the closed ball B such that $Tu_k \to w$. (We denote strong convergence by $\to$ as usual, weak* convergence in V^* or weak convergence in V by $\rightharpoonup$.) Since B is weak* compact, we may assume that $u_k \rightharpoonup u$, $u \in B$. It suffices to show that $Tu = w$. Since C is compact, we may assume also that $C(u_k) \to w_1$. Then $S(u_k, u_k) = T(u_k) - C(u_k) \to w - w_1$, and by the complete continuity of S in its second variable, $S(u, u_k) \to S(u, y)$. Hence

$$(S(u_k, u_k) - S(u, u_k), u_k - u) \to (w - w_1 - S(u, u), u - u) = 0.$$

Applying condition (4), we have $C(u_k) \rightharpoonup C(u)$, i.e. $C(u) = w_1$. Finally for any v in V^*, we have for every $k \geqq 1$,

$$(S(u_k, u_k) - S(v, u_k), u_k - v) \geqq 0.$$

As $k \to +\infty$, $S(u_k, u_k) \to w - C(u)$, $S(v, u_k) \to S(v, u)$, and $u_k - v \rightharpoonup u - v$. Hence

$$(S(u_k, u_k) - S(v, u_k), u_k - v) \to (w - C(u) - S(v, u), u - v) \geqq 0.$$

Applying our earlier remark with $Q(u) = S(v, u)$, we see that $S(u, u) = w - C(u)$, i.e. $T(u) = w$, and $T(B)$ is closed. q.e.d.

LEMMA. *For each $\lambda > 0$, let $T_\lambda = T + \lambda M$, where M is the mapping given in the definition of the improvability of V. To prove Theorem 2, it suffices to show that $0 \in R(T_\lambda)$, for each $\lambda > 0$.*

PROOF OF THE LEMMA. There exists a function $c(r)$ on R^1 with $c(r) \to +\infty$ as $r \to +\infty$ such that $(Tu, u) \geq c(\|u\|)\|u\|$. For each $\lambda > 0$, $(T_\lambda u, u) = (Tu, u) + \lambda(Mu, u) \geq (Tu, u)$ for $\|u\| \geq R$, independent of λ. If $T_\lambda u_\lambda = 0$, it follows that either $\|u_\lambda\| \leq R$ or $c(\|u_\lambda\|) \leq 0$. Hence $\|u_\lambda\|$ is uniformly bounded. Hence $M(u_\lambda)$ is uniformly bounded. Choosing a sequence of $\lambda_k \to 0$, we see that $T(u_\lambda) = T_\lambda(u_\lambda) - \lambda M(u_\lambda) = -\lambda M(u_\lambda)$ $\to 0$. Since the u_λ all lie in a fixed ball B, 0 lies in the closure of $T(B)$. By condition (4)′, 0 lies in $T(B)$. q.e.d.

PROOF OF THEOREM 2 COMPLETED. Let $\lambda > 0$ be fixed. We wish to show that 0 lies in $R(T_\lambda)$. We construct a one-parameter family of mappings K_t of V^* into V^* for $0 \leq t \leq 1$, in the following way: For $t \in [0, 1]$, let

$$M_t(v, u) = (1 - t)\{S(v, u) + C(u) + \lambda M(v)\} + tM(v).$$

For each fixed t and u, $M_t(\cdot, u)$ is a continuous monotone mapping of V^* into V, with $(M_t(v, u) - M_t(v_1, u), v - v_1) \geq \lambda(M(v) - M(v_1), v - v_1)$ and

$$(M_t(v, u), v) \geq (M_t(v, u) - M_t(0, u), v) + (M_t(0, u), v)$$
$$\geq \lambda(M(v), v) - g(\|u\|)v.$$

Hence $(M_t(v, u), v)/\|v\| \to +\infty$ as $\|v\| \to +\infty$ (uniformly for bounded u) and by Theorem 5 of [4], there exists an unique element $v = K_t(u)$ for which

$$M_t(v, u) = 0.$$

Since the fixed points of the mapping K_0 are the zero points of T_λ, it suffices by the Leray-Schauder theorem [16] to verify the following:

(I) K_1 is a constant map. There exists a constant M independent of t such that if $K_t(u) = u$ for any t in $[0, 1]$, then $\|u\| \leq M$.

(II) The map $(t, u) \to K_t(u)$ is a continuous map of $[0, 1] \times V^*$ into V^* such that for any bounded set B, the image of $[0, 1] \times B$ is precompact.

PROOF OF (I). Since $M_1(v, u) = M(v)$, K_1 is a constant map. If $K_t(u) = u$, we have $M_t(u, u) = 0$. However,

$$(M_t(u, u), u) = (1 - t)(Tu, u) + \{(1 - t)\lambda + t\}(Mu, u) \geq \lambda(M(u), u)$$

implies that in that case $(M(u), u) \leq 0$, and $\|u\| \leq M$ with a constant M independent of t. q.e.d.

PROOF OF (II). It suffices to prove both of the following:

(a) If $u_j \to u$, $v_j = K_{t_j} u_j \to v$, $t_j \to t$, then $v = K_t u$.

(b) If $u_j \to u$, $t_j \to t$, $v_j = K_{t_j} u_j \to v$, then for a subsequence, $v_j \to v$.

PROOF OF (a). We have $C(u_j) \to C(u)$, and for any z in V^*, $S(z, u_j)$ $\to S(z, u)$. Since $M_{t_j}(v_j, u_j) = 0$, if we set $R_t(v, u) = (1-t)S(v, u)$ $\{(1-t)\lambda + t\} M(v)$, we see that $R_{t_j}(v_j, u_j) = M_{t_j}(v_j, u_j) - (1-t_j)C(u_j)$ $\to (1-t)C(u)$. Since

$$(R_{t_j}(v_j, u_j) - R_{t_j}(z, u_j), v_j - z) \geqq 0,$$

passing to the limit as $j \to +\infty$, we obtain

$$(-(1 - t)C(u) - R_t(z, u), v - z) \geqq 0.$$

Using our previous remark with $Q(z) = R_t(z, u)$, we see that $R_t(v, u)$ $= -(1-t)C(u)$, i.e. $M_t(v, u) = 0$ and $v = K_t(u)$. q.e.d.

PROOF OF (b). Since C is compact and $\{u_j\}$ is bounded, we may assume by passing to a subsequence that $C(u_j) \to w_1$. Using the mapping R_t defined in the proof of (a), we have

$$R_{t_j}(v_j, u_j) \to - (1 - t)w_1.$$

Since

$$(R_{t_j}(v, u_j) - R_{t_j}(v, u_j), v_j - v) \geqq \lambda(M(v) - M(v), v_j - v) \geqq 0$$

and

$$(R_{t_j}(v_j, u_j) - R_{t_j}(v, u_j), v_j - v) \to (-(1 - t)w_1 - R_t(v, u), 0) = 0$$

it follows that $(M(v_j) - M(v), v_j - v) \to 0$ and hence $v_j \to v$. q.e.d.

BIBLIOGRAPHY

1. F. E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc. **69** (1963), 862–874.

2. ———, *Strongly nonlinear parabolic boundary value problems*, Amer. J. Math. **86** (1964), 339–357.

3. ———, *Nonlinear elliptic problems*. II, Bull. Amer. Math. Soc. **70** (1964) 299–301.

4. ———, *Nonlinear elliptic boundary value problems*. II, Trans. Amer. Math. Soc. **117** (1965), 530–550.

5. ———, *Nonlinear equations of evolution*, Ann. of Math. **80** (1964), 485–523.

6. ———, *Multivalued monotone nonlinear mappings and duality mappings in Banach spaces*, Trans. Amer. Math. Soc. **118** (1965), 338–351.

7. ———, *Nonlinear initial value problems*, Ann. of Math. **81** (1965), 51–87.

8. ———, *Remarks on nonlinear functional equations*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 985–989.

9. ———, *Existence and uniqueness theorems for solutions of nonlinear boundary problems*, Proc. Sympos. Appl. Math. Vol. 17, Amer. Math. Soc., Providence, R. I., 1965, pp. 24–49.

10. ———, *Nonlinear monotone operators and convex sets in Banach spaces*, Bull. Amer. Math. Soc. **71** (1965), 780–785.

11. ――――, *Nonlinear operators in Banach spaces and nonlinear boundary value problems*, (to appear).

12. U. Dubinski, *Some integral inequalities and solvability of degenerate quasilinear elliptic systems*, Mat. Sb. **64** (1964), 458–480.

13. P. Hartman and G. Stampacchia, *On some nonlinear elliptic differential-functional equations*, (to appear).

14. M. A. Krasnoselski and Ya. B. Rutickii, *Convex functions and Orlicz spaces*, Moscow, 1958. (Transl. by Noordhoff, Groningen, 1961.)

15. J. Leray and J. L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques nonlinéaire par le méthode de Minty-Browder*, Bull. Soc. Math. France **93** (1965), 97–107.

16. J. Leray and J. Schauder, *Topologie et équations fonctionelles*, Ann. Sci. École Norm. Sup. (3) **51** (1934) 45–78.

17. G. J. Minty, *On a "monotonicity" method for the solution of nonlinear equations in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1038–1041.

18. M. I. Višik, *Quasi-linear strongly elliptic systems of differential equations in divergence form*, Trudy Moscow. Mat. Obšč. **12** (1963), 125–184.

19. ――――, *Solvability of the first boundary value problem for quasilinear equations with rapidly increasing coefficients in Orlicz classes*, Dokl. Akad. Nauk. SSSR **151** (1963), 758–761.

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