EXTENSION OF NONLINEAR CONTRACTIONS

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The following problem was suggested as a research problem by R. A. Hirschfeld in Bull. Amer. Math. Soc. **71** (1965), 495:

E and F are Banach spaces, F reflexive, D is a subset of E and $T: D \to F$ a nonlinear contraction, i.e. $\|Tx_1 - Tx_2\| \leqq \|x_1 - x_2\|$ whenever $x_1, x_2 \in D$. Can T be extended to a contraction $T': E \to F$?

Hirschfeld observes that the answer is "yes" when $E = F =$ Hilbert space.

The following simple example shows that the answer is "no" in general. In the two-dimensional plane R^2 consider a regular hexagon H, with its center at the origin, and a circle C inscribed in H. Let E and F be R^2 equipped with norms $\|\cdot\|_E$ and $\|\cdot\|_F$ defined by $\{x; \|x\|_E = 1\} = H$, $\{x; \|x\|_F = 1\} = C$. Let x_1 and x_2 be two consecutive points of contact between H and C. Then

$$\|x_1\|_E = \|x_1\|_F = \|x_2\|_E = \|x_2\|_F = \|x_1 - x_2\|_E = \|x_1 - x_2\|_F = 1$$

so that if $D = \{0, x_1, x_2\}$ and $T(0) = 0$, $Tx_1 = x_1$, $Tx_2 = x_2$, T is a contraction of D into F. Now, if $z = (x_1 + x_2)/3$, it is easily seen that $\|z\|_E = \|z - x_1\|_E = \|z - x_2\|_E = 1/2$. Hence, if T could be extended to a contraction $T': E \to F$, then the point $u = T'z$ would satisfy

$$\|u\|_F \leqq 1/2, \qquad \|u - x_1\|_F \leqq 1/2, \qquad \|u - x_2\|_F \leqq 1/2$$

which is clearly impossible.

We have, however, been able to prove some positive results. In order to state these results, we introduce the following terminology. If E and F are normed linear spaces, we say that (E, F) has the contraction-extension (c.e.) property if: for any subset $D \subset E$ and any contraction $T: D \to F$ there is an extension of T to a contraction $T': E \to F$.

We then have

THEOREM 1. *If E and F are real or complex Banach spaces, if F is strictly convex and if (E, F) has the c.e. property, then E and F are Hilbert spaces.*

OUTLINE OF PROOF. It is clearly sufficient to assume that E and F are real spaces. Using the strict convexity of F, it is then easy to show that, if $x, y \in E$, $u, v \in F$ and if $\|x\| = \|u\|$, $\|y\| = \|v\|$, $\|x - y\| = \|u - v\|$, then $\|ax + by\| \geqq \|au + bv\|$ for all real numbers a, b.

If x and y are elements or a real normed linear space, we say that x is normal to y if $\|x+ay\| \geqq \|x\|$ for all real numbers a, and then we write xNy. Using our above result and a limiting process we may prove: if $x, y \in E$, $u, v \in F$ and if $\|x\| = \|u\|$, $\|y\| = \|v\|$, xNy, uNv, then $\|ax+by\| \geqq \|au+bv\|$ for all a, b.

With the aid of this result, it is now possible to show that normality is a symmetric relation in both E and F. Day [2] has given a construction of all two-dimensional spaces with symmetry of normality. By means of this construction and our previous results we may conclude: if $x, y \in E$, $u, v \in F$ and if $\|x\| = \|u\|$, $\|y\| = \|v\|$, xNy, uNv, then $\|ax+by\| = \|au+bv\|$ for all a, b. This implies that both E and F have the following property, formulated for a normed linear space L:

There is a single-valued function f of two real variables so that for any $x, y \in L$ such that xNy we have $\|x+y\| = f(\|x\|, \|y\|)$.

But this property is characteristic of euclidean (i.e. prehilbert) spaces, as can be shown in a number of ways. (See for instance Hopf [4], where this is shown even without assuming symmetry of the norm.)

THEOREM 2. *The following two properties of a real Banach space F are equivalent*:

(i) *(E, F) has the c.e. property for every real Banach space E*

(ii) *any family of closed spheres in F, such that any two members of it intersect, has a nonempty intersection.*

OUTLINE OF PROOF. (i)$\Rightarrow$(ii) is proved by first observing that, for any set S, the Banach space $m(S)$ of all bounded real-valued functions on S with the supremum norm, has property (ii). We then embed F isometrically in a suitable $m(S)$. If (S_i), $i \in I$, are closed spheres in F such that $S_i \cap S_j \neq \emptyset$ for all i, j, then for the corresponding spheres $\sum_i$ in $m(S)$ we have $\bigcap_i \sum_i \neq \emptyset$. Using the c.e. property of $(m(S), F)$ we then conclude that $\bigcap_i S_i \neq \emptyset$.

(ii)$\Rightarrow$(i) is proved by Zorn's lemma in a straightforward way.

Theorem 2 shows the intimate connection between our present problem and the problem of linear, norm-preserving extension of continuous linear transformations. In fact, it has been proved by Nachbin [6] that property (ii) for a real Banach space F is equivalent to

(iii) for any real Banach space E, any closed linear subspace S of E and any continuous linear transformation T of S into F, there exists a linear extension T' of T to E with values in F and $\|T'\| = \|T\|$.

Moreover, through the work of Aronszajn-Panitchpakdi [1], Goodner [3], Kelley [5] and Nachbin, it is also known that a real space F has property (iii) if and only if F is linearly isometric to a space $C(S)$, the space of real-valued continuous functions on a compact, Hausdorff and extremally disconnected space S. (For a survey of these and related problems, see Nachbin [7].)

Thus we have the following

COROLLARY TO THEOREM 2. *If F is a real Banach space, then (E, F) has the c.e. property for every real Banach space E if and only if F is linearly isometric to a space $C(S)$, where S is compact, Hausdorff and extremally disconnected.*

Finally, using the corollary it is easy to show that a complex Banach space F can never have property (ii). Hence we may conclude that there is no complex Banach space F such that (E, F) has the c.e. property for every complex Banach space E.

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