A CHARACTERIZATION OF THE EUCLIDEAN SPHERE

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1. **Introduction.** Let M be a connected Riemannian manifold of dimension n, $C_0(M)$ its largest connected group of conformal transformations and $I_0(M)$ its largest connected group of isometries. In an earlier paper [2], one of the authors and S. Kobayashi established the following result:

THEOREM 1. *A compact homogeneous Riemannian manifold for which $C_0(M)\neq I_0(M)$ and $n>3$ is globally isometric with a sphere.*[2]

In the final step of the proof of this theorem the following statement, which is by no means easy to establish, was utilized:

PROPOSITION 1 (YANO-NAGANO [6]). *A complete Einstein space for which $C_0(M)\neq I_0(M)$ and $n>2$ is globally isometric with a sphere.*

Without this fact it was shown that *the simply connected Riemannian covering of M is globally isometric with a sphere.* Using this statement, an elementary proof of Theorem 1, i.e. a proof which does not use Proposition 1, is given (see Proposition 4).

All other results in this direction employ Proposition 1 in the final analysis. We list several of these:

PROPOSITION 2 (NAGANO [4]). *A complete Riemannian manifold with parallel Ricci tensor for which $C_0(M)\neq I_0(M)$ and $n>2$ is globally isometric with a sphere.*

This generalizes Proposition 1.

PROPOSITION 3 (LICHNEROWICZ [3]). *Let M be a compact Riemannian manifold of dimension $n>2$ whose scalar curvature R is a positive constant and for which* trace $Q^2=$const. *where Q is the Ricci operator (see* [1, *p.* 87]). *Then, if $C_0(M)\neq I_0(M)$, M is globally isometric with a sphere.*

This generalizes Theorem 1 and Proposition 2.

In §4, Proposition 1 will be generalized. Denote the Lie algebra of

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[2] The first part of the proof of Theorem 1 appears in a previous paper published in the Amer. J. Math 84 (1962), 170–174 by S. I. Goldberg and S. Kobayashi entitled *The conformal transformation group of a compact Riemannian manifold.*

$C_0(M)$ by $C_0(M)$. Let $X \in C_0(M)$ and ξ be the covariant form of X defined by duality by the Riemannian metric $\langle\ ,\ \rangle$ of $M: \xi = \langle X, \cdot \rangle$. Let $C_0^*(M) = \{\xi \mid \xi = \langle X, \cdot \rangle,\ X \in \mathbf{C}_0(M)\}$ and denote by d and δ the differential and codifferential operators of de Rham and Hodge. Then [cf. M. Obata and K. Yano, C. R. Acad. Sci. Paris **260** (1965), 2698–2700].

THEOREM 2. *Let M be a compact Riemannian manifold of dimension $n > 3$ for which $R = \text{const.}$ and $C_0(M) \neq I_0(M)$. If $d\delta C_0^*(M)$ is an invariant subspace of Q, then M is globally isometric with a sphere.*

This theoremal most completely answers the question raised in [2], namely,

Is a compact manifold of dimension $n > 2$ with constant (positive) scalar curvature for which $C_0(M) \neq I_0(M)$ isometric with a sphere?

Observe that Proposition 1 is an easy consequence of Theorem 2.

2. **Isometries and conformal fields.** If T is an isometry of the unit sphere S^n in E^{n+1}, then T may be viewed as an orthogonal linear transformation of E^{n+1} restricted to S^n. It is clear that any such isometry will map Killing fields into Killing fields and constant conformal fields ($d\phi = \langle X, \cdot \rangle$) into constant conformal fields. Thus if a conformal field is invariant under T so are its constant and Killing parts. It follows that if T leaves a non-Killing conformal field invariant then it has a fixed point, namely $N/\|N\| \in S^n$, where N is a constant field in E^{n+1} and $N - \langle N, P \rangle P$ ($P \in S^n$) is the constant part of V.

3. **Conformal fields on a manifold of positive constant curvature.** If M is a compact Riemannian manifold with constant positive curvature then the nature of the conformal group of M does not change if we normalized the curvature so that it is 1. Thus, S^n is the simply connected covering Riemannian manifold of M. If M has a non-Killing conformal vector field V then this vector field may be lifted to a non-Killing conformal vector field $\overline{V}$ on S^n. Moreover, $\overline{V}$ is invariant by the deck transformations of the covering space $S^n \to M$. But only the identity deck transformation can have a fixed point, and since a deck transformation is an isometry we have from §2 that there are no deck transformations except the identity. This proves the following special case of Proposition 1:

PROPOSITION 4. *If a compact Riemannian manifold of positive constant curvature admits a non-Killing conformal vector field then it is globally isometric with a sphere.*

Since the above argument clearly works for $n = 2$, we have

COROLLARY. *The real projective plane does not admit a non-Killing conformal vector field.*

4. Conformal fields on manifolds of constant scalar curvature. We sketch the proof of Theorem 2. Let $\xi = d\phi$ be an element of $C_0^*(M)$. Then, $Qd\delta\xi = d\delta Q\xi$. Conversely, suppose $d\delta C_0^*(M)$ is an invariant subspace of Q. Then, there exists a $\xi \in C_0^*(M)$ such that $d\delta\xi$ is an eigenvector of Q, that is $Qd\delta\xi = (R/n)d\delta\xi \in C_0^*(M)$. Moreover, since $\Delta\delta\xi = (R/(n-1))\delta\xi$ (see [1, p. 264]),

$$d\delta\xi = \frac{R}{n-1}\xi + \langle Y, \cdot \rangle$$

where Y is a Killing field. That this can only hold if M has constant curvature is a consequence of the following:

LEMMA. *Let M be a compact Riemannian manifold on which there is a nonconstant function ϕ: $M \rightarrow R$ whose gradient $\xi = d\phi \in C_0^*(M)$. Then, there are no nonzero tensors of the type (r, s), $0 < 2(s-r) \leqq n$ invariant under X where $\xi = \langle X, \cdot \rangle$.*

The proof of this lemma is intended for a subsequent paper.

Setting

$$T(A, B) = \tilde{R}(A, B) - \frac{R}{n}\langle A, B \rangle,$$

where $\tilde{R}$ is the Ricci tensor, it can be shown that $\theta(\xi)T = 0$. Since the Weyl conformal curvature tensor is invariant under X, we see by the lemma that M is conformally flat. However, since $\theta(\xi)T$ vanishes, a further application of the lemma gives $T = 0$, that is M is an Einstein space. But a conformally flat Einstein space has constant curvature, and so by Proposition 4, M is globally isometric with a sphere. This proves Theorem 2 and generalizes Proposition 1.

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