THE TOPOLOGICAL COMPLEMENTATION PROBLEM

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Let Σ be the lattice of all topologies definable on an arbitrary set E. Then Σ is a complete lattice with the trivial topology, $\{\emptyset, E\}$, as the least element and the discrete topology, $P(E)$, as the greatest element.

The problem of complementation in the lattice Σ has been outstanding for some time although several investigators have provided partial solutions. Hartmanis [6] first showed that Σ was a complemented lattice if the set E was finite and Gaifman [4] proved Σ was complemented if E was countable. Berri [1], using the results of Gaifman, was able to provide complements for certain special topologies such as a topological group with a dense, nonopen, countable subgroup.

It is the purpose of this paper to introduce the lattice of principal topologies, and to establish that the lattice Σ of all topologies on a set E is complemented.

The following theorems are stated without proof. The full details will be published elsewhere.

1. **Principal topologies.** A topology $\tau \in \Sigma$ is called an ultraspace if the only topology finer than τ is the discrete topology. Fröhlich [3] shows that every topology τ is the infimum of ultraspaces finer than τ. For a filter $\mathfrak{F}$ on E and a point $x \in E$, Fröhlich defined $\mathfrak{S}(x, \mathfrak{F})$ to be the family of sets $P(E-\{x\}) \cup \mathfrak{F}$, which is a topology. He proved the ultraspaces are the topologies of the form $\mathfrak{S}(x, \mathfrak{U})$ where $x \in E$ and $\mathfrak{U}$ is an ultrafilter on E different from the principal ultrafilter at x, $\mathfrak{U}(x)$. The set of ultraspaces may be divided into two classes each of which generates a sublattice of Σ. One of these sublattices consists of all T_1-topologies. The other is called the lattice of principal topologies.

Every topology τ on E is the infimum of all ultraspaces on E finer than τ. If also $\tau = \inf\{\mathfrak{S}(x, \mathfrak{U}(y)) \mid \mathfrak{S}(x, \mathfrak{U}(y)) \geqq \tau\}$ then τ will be called a principal topology.

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THEOREM 1.1. *The principal topologies form a sublattice of the lattice* Σ.

The lattice of principal topologies on E will be denoted Π. The lattice Π is a complete lattice but is not a complete sublattice of Σ.

THEOREM 1.2. *A topology* τ *on* E *is a principal topology if and only if for each* $x \in E$ *there is a minimal element in* τ *containing* x.

A principal topology then, is one which is closed under arbitrary intersections.

The family $\mathcal{G}$ of pre-order relations forms a complete lattice with $E \times E$ as the greatest element and $\Delta = \{(x, x) \mid x \in E\}$ as the least element.

A relation G defines a topology τ_G on E: A set $S \subset E$ is open if and only if for each $x \in S$, if $(x, y) \in G$ then $y \in S$. The topology τ_G determined by the relation G is a principal topology.

THEOREM 1.3. *There is a one-to-one correspondence between principal topologies in* Π *and pre-order relations in* $\mathcal{G}$.

THEOREM 1.4. *The lattice* Π *of principal topologies is anti-isomorphic to the lattice* $\mathcal{G}$ *of pre-order relations.*

THEOREM 1.5. *The lattice* $\mathcal{G}$ *of pre-order relations on a set* E *is a complemented lattice.*

It follows from Theorem 1.4 that Π is a complemented lattice. Since every topology on a finite set is a principal topology, this result extends the result of Hartmanis [6].

2. **Complementation of the lattice Σ.** A topology $\tau \in \Sigma$ has a principal complement if it has a lattice complement which is a principal topology.

Gaifman [5] proved that if every T_1-topology on a set has a complement then every topology on that set has a complement. Modifying his proof we obtain

THEOREM 2.1. *If every* T_1*-topology on a set* E *has a principal complement, then every topology on* E *has a principal complement.*

THEOREM 2.2. *Let* τ *be a topology on a set* $E = E_1 \cup E_2$ *where* $E_1 \cap E_2 = \emptyset$, *such that* $\tau \mid E_1$ *and* $\tau \mid E_2$ *have principal lattice complements. Then* τ *has a principal complement.*

THEOREM 2.3. *If every topology* (T_1*-topology*) *with no isolated points has a principal complement, then every topology* (T_1*-topology*) *has a principal complement.*

The next theorem is an extension of a result of Berri [2]: A topology on a set E has a complement if there is a decomposition of E into countable sets such that no union of any proper subcollection is open.

THEOREM 2.4. *Let τ be a topology on a set E such that*
(i) $E = \bigcup_{\alpha\in\theta} E_\alpha$, *where E_α's are pairwise disjoint,*
(ii) $\tau | E_\alpha$ *has a principal complement* τ'_α, *for all* $\alpha \in \theta$,
(iii) *if* $V \in \tau$ *and if* $V \neq E, \emptyset$, *then V is not the union of E_α's.*
Then τ has a principal complement τ'. If some τ'_α has an isolated point, so does τ'.

THEOREM 2.5. *Let τ be a T_1-topology on a set E containing a proper open set S with at least two points, such that $\tau | S$ has a principal complement with an isolated point. Then τ has a principal complement with an isolated point.*

THEOREM 2.6. *A T_1-topology with no isolated points has a principal complement with an isolated point.*

Now from Theorems 2.6, 2.3 and 2.1 we have

THEOREM 2.7. *The lattice of topologies on any set is complemented. Moreover, each topology has a principal complement.*

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