REMARK ON THE MODULUS OF COMPACT OPERATORS

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If X and Y are Banach lattices (see Day [1]) the linear continuous operators T from X to Y are partially ordered by: $T_1 \geqq T_2$ if and only if $T_1 f \geqq T_2 f$ for all $0 \leqq f \in X$. For some kinds of pairs (X, Y), e.g. $X = Y = L_1$ or L_∞, the continuous operators have been shown to form a Banach lattice (see Kantorovitch [2]). This note contains a surprising example, showing that the modulus of a compact operator need *not* necessarily be compact, and a sufficient condition under which the modulus will be compact.

EXAMPLE. We shall modify an example in [3]: Let Ω be the union of disjoint sets Ω_n $(n = 1, 2, \cdots)$ where Ω_n consists of 2^n points x_{ni} $(i = 1, \cdots, 2^n)$, with measure 1 each. Define an infinite matrix $A = (a_{ik})$ by induction:

$$(0) \qquad A_1 = \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}, \qquad A_{j+1} = \begin{pmatrix} +A_j & +A_j \\ +A_j & -A_j \end{pmatrix}$$

where A_n is the matrix of the first 2^n rows and columns in A. If $\chi_{\{x\}}$ is the characteristic function of $\{x\}$ we define the operator S_n in $L_2(\Omega)$ by:

$$(1) \qquad \chi_{\{x_{nk}\}} S_n = 2^{-n} \sum_{i=1}^{2n} a_{ik} \chi_{\{x_{ni}\}} \quad \text{and} \quad \chi_{\{x_{mk}\}} S_n = 0 \quad \text{for } m \neq n.$$

$|S_n|$ is obtained by using $|a_{ik}| = 1$ instead of a_{ik} in (1). In [3, p. 171] the operators $T_n = 2^{n/2} S_n$ were investigated and the norms observed to be $\|T_n\| = 1$, $\| \, |T_n| \, \| = 2^{n/2}$. $S = \sum_{n=1}^{\infty} S_n$ is a continuous operator with $|S| = \sum_{n=1}^{\infty} |S_n|$. Since for each N, $\sum_{n=1}^{N} S_n$ is a compact operator and these tend to S in norm, S is compact. To see that $|S|$ is not compact look at the functions f_n which are $= 2^{-n/2}$ on Ω_n and 0 elsewhere. They satisfy $f_n = |S| f_n$ and $\|f_n\| = 1$.

If the modulus $|T|$ of an operator T exists, it has the form

$$|T| f = \sup_{|g| \leqq f} |Tg| \qquad \text{for } f \in X^+ = \{h \in X : h \geqq 0\}$$

(see [3]).[2] In [3] it has been shown, that the modulus of any compact

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[2] In [3] the definition of a Banach lattice unnecessarily is slightly more special than in [1].

operator $T: X \rightarrow Y$ is compact, if X is an L-space (see Day [1]) and the monotone convergence theorem holds in Y.

THEOREM. *If X is any Banach lattice and Y is an M-space (see [1]) the compact operators T from X to Y form a Banach lattice.*

The proof is based on the following lemma:

LEMMA. *If Y is a M-space and $C \subseteq Y$ is conditionally compact, $\sup\{f : f \in A\}$ exists for all $A \subseteq C$ and the set of all such suprema is conditionally compact.*

The lemma is proved by representing Y as a subspace of the space $C(\Omega)$ of continuous functions on some compact space Ω and by an application of the Arzela-Ascoli theorem.

REFERENCES

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2. L. Kantorovitch, *Linear operations in semi-ordered spaces*, Mat. Sb. (N.S.) **49** (1940), 209–284.

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