ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR VOLTERRA EQUATIONS

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In this note we show how certain known results for delay differential equations can be extended to systems of integral equations of the form

$$(1) \qquad x(t) = f(t) + \int_0^t a(t-s)g(s, x(s))\, ds \qquad (t \geqq 0).$$

We make the following assumptions:

(A1) $f(t)$ is uniformly continuous and bounded on $0 \leqq t < \infty$,

(A2) $a(t)$ is a square matrix whose entries are $L_1(0, \infty)$,

(A3) $g(t, x)$ is continuous in (t, x) for $0 \leqq t < \infty$, $|x| < \infty$ and g is uniformly almost periodic in t uniformly on compact subsets of x in real n-space R^n, and

(A4) $x(t)$ is a bounded solution of (1) for $0 \leqq t < \infty$.

Let Ω be the positive limit set of $x(t)$. We refer to [2] for the definitions and properties of almost periodic functions and limit sets. The analog for integral equations of [2, Theorem 1] is

THEOREM 1. *If* (A1)–(A4) *are satisfied, then to each point z in Ω there corresponds a sequence $t_m \to \infty$ as $m \to \infty$ and functions $G(t, x)$, $X(t)$ and $F(t)$ such that*

(i) $\text{limit}_{m\to\infty} |x(t+t_m) - X(t)| + |f(t+t_m) - F(t)| = 0$ *uniformly on compact subsets of* $-\infty < t < \infty$,

(ii) $\text{limit}_{m\to\infty}\, g(t+t_m, x) = G(t, x)$ *uniformly for all t and for x on compact sets, and*

(iii) *on the interval* $-\infty < t < \infty$, $X(t) \in \Omega$ *and*

$$(2) \qquad X(t) = F(t) + \int_{-\infty}^t a(t-s)G(s, X(s))\, ds.$$

PROOF. As is well known in harmonic analysis the convolution of an L_1 function with an essentially bounded function yields a uniformly continuous function. Hence $x(t)$ is bounded and uniformly continuous on the interval $0 \leqq t < \infty$.

Given z in Ω let $\{t_m\}$ be a sequence such that $t_m \to \infty$ and $x(t_m) \to z$ as $m \to \infty$. Define $x_m(t) = x(t+t_m)$ and $f_m(t) = f(t+t_m)$ for $t \geqq -t_m$. Since

$$x_m(t) = f_m(t) + \int_{-t_m}^t a(t-s)g(s+t_m, x_m(s))\, ds,$$

the proof can now be completed in the same way as the proof of [2, Theorem 1].

We remark that with essentially the same proof one can establish a modified version of Theorem 1 in which the lower limit of integration in equation (1) is $-\infty$. Note also that one could add to the right side of (1) a bounded measurable function $h(t)\to 0$ as $t\to\infty$.

Since a bounded continuous function must tend to its positive limit set, Theorem 1 above can sometimes be used to obtain results on the asymptotic behavior of solutions. We shall illustrate the technique with some examples. Consider the scalar equation

$$x(t) = f(t) - \int_0^t a(t-s)x(s)\,ds. \tag{3}$$

Paley and Wiener [4, pp. 58–63] prove:

THEOREM 2 (PALEY-WIENER). *Suppose* $a(t)$ *is* $L_1(0, \infty)$ *and* $f(t)$ *is bounded, measurable and tends to a limit* f_0 *as* $t\to\infty$. *For each such* f *the solution of* (2.1) *is bounded and tends to the limit*

$$x(t) \to x_0 = f_0 \Big/ \left(1 + \int_0^\infty a(s)\,ds\right) \quad \text{as} \quad t\to\infty \tag{4}$$

if and only if when $\mathrm{Re}(u) \geqq 0$ *one has*

$$\int_0^\infty a(t)\exp(-ut)\,dt \neq -1. \tag{5}$$

To this we add

COROLLARY 1. *Let* $a(t)$ *and* $f(t)$ *be as in Theorem* 2. *All bounded solutions of* (3) *satisfy* (4) *if and only if* (5) *holds whenever* $\mathrm{Re}(u)=0$.

Under the hypothesis of Corollary 1 some solutions may be unbounded as $t\to\infty$. If we do have a bounded solution, then Theorem 1 above applies. The limiting system corresponding to (2) is in this case

$$X(t) = f_0 - \int_{-\infty}^t a(t-s)X(s)\,ds.$$

The transformation $Y(t) = X(t) - x_0$ gives

$$Y(t) = -\int_{-\infty}^t a(t-s)Y(s)ds \qquad (-\infty < t < \infty).$$

For this last equation it is known that $Y(t)\equiv 0$ is the only bounded solution if and only if the Fourier transform of $a(t)$ is never -1, cf. [4, p. 59 and p. 63].

Levin [1] has proved a nonlinear version of Theorem 2. Consider

$$x(t) = f(t) - \int_0^t a(t - s)g(x(s))\, ds, \tag{6}$$

with the following assumptions:

(B1) f is bounded and measurable on $0 \leqq t < x$ and tends to f_0 as $t \to \infty$,

(B2) $g(x)$ is $C(-\infty, \infty)$, $g(0) = 0$, and g is strictly increasing, and

(B3) $a(t)$ is $C[0, \infty)$, $C^1(0, \infty)$ and $L_1(0, \infty)$, $a(t) \geqq 0$, $a'(t) \leqq 0$ and $a'(t) \not\equiv 0$ on any interval except possibly $a''(t) \equiv 0$ for all large t.

It is possible to separate the boundedness criterion in Levin's problem in the same way that Corollary 1 refines Theorem 2. The limiting system for (6) is

$$X(t) = f_0 - \int_{-\infty}^t a(t - s)g(X(s))\, ds. \tag{7}$$

Assumptions (B2) and (B3) insure that (7) has a unique *constant* solution x_0. Moreover, one can show that $x(t) \equiv x_0$ is the only bounded solution of (7). This proves

THEOREM 3. *If* (B1)–(B3) *hold and if $x(t)$ is a bounded solution of* (6), *then $x(t) \to x_0$ as $t \to \infty$.*

From Levin's results we see that if, in addition, $f'(t)$ exists and is $L_1(0, \infty)$, then all solutions of (6) exist and are bounded for positive t. Other criterion can be given for boundedness. For example suppose $f(t)$ is bounded, $a(t)$ is $L_1(0, \infty)$ with $a(t) \geqq 0$ almost everywhere and $g(x) = \exp(x) - 1$. If $x(t)$ is a solution of (6), then for as long as it exists

$$x(t) \geqq f(t) - \int_0^t a(t - s)\, ds > -M.$$

Hence we also have

$$x(t) \leqq f(t) + \int_0^t a(t - s)g(-M)\, ds < N.$$

By general results of Nohel [3], $x(t)$ exists and is bounded on the interval $0 \leqq t < \infty$.

References

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