ON CERTAIN BISIMPLE INVERSE SEMIGROUPS[1]

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If S is a semigroup, E_S will denote the collection of idempotents of S. A bisimple semigroup S is called I-bisimple if and only if $E_S = \{e_i : i \in I, \text{ the integers}\}$ with $e_i \leqq e_j$ if and only if $i \geqq j$. We announce the determination of the structure of I-bisimple semigroups mod groups and a determination of several of their properties. We also give a certain generalization of the bicyclic semigroup and indicate an application of this result. We use the notation and terminology of [2].

THEOREM 1. *S is an I-bisimple semigroup if and only if $S \cong GXIXI$ under the multiplication*

$$(1) \qquad (g, n, m)(h, p, q) = (g\alpha^{p-r}h\alpha^{m-r}, n + p - r, m + q - r)$$

where $r = \min(m, p)$, α is an endomorphism of G, and α^0 is the identity transformation or equivalently

$$(g, n, m)(h, p, q) = (g\alpha^{s-m-p}h\alpha^{s-q}, n + p, s)$$

where $s = \max(m+p, q)$.

PROOF. [9, Theorem], [1, Main Theorem], [8, Theorem 1.2 and Theorem 2.2] and [5, Theorem 3.3] are important.

REMARK. An I-bisimple semigroup S has no identity and hence its structure may not be obtained by specializing the Clifford structure theorem [1]. S is a union of a chain of bisimple (inverse) semigroups S_i $(i \in I)$ with identity such that $E_{S_i} = \{e_i : i \in I^0, \text{ the non-negative integers}\}$ with $e_i \leqq e_j$ if and only if $i \geqq j$.[2] The structure of these semigroups was given mod groups by Reilly [6] and Warne [11]. Warne obtained the result by specializing the Clifford structure theorem [1]. Incidently, the multiplication is given by (1) with I^0 replaced for I.

If S is an I-bisimple semigroup with structure group G and structure endomorphism α, we will write $S = (G, \alpha)$.

Let N denote the natural numbers.

THEOREM 2. *Let $S = (G, \alpha)$ and $S^* = (G^*, \beta)$. Let $\{f_i : i \in I \backslash N\}$ be a*

[1] These structure theorems represent a next stage in the development of bisimple semigroups to the Rees Theorem in that the determination is complete (mod groups).

[2] The structure of bisimple (inverse) semigroups such that E_S is linearly ordered has been given mod bisimple inverse semigroups with identity by Warne [9].

collection of homomorphisms of G into G^*, $\{X_i : i \in I \backslash N\}$ *be a collection of nondecreasing functions of I into I*, $a \in I^0$, *and* $\{z_i : i \in I \backslash N\}$ *be a collection of elements of* G^* *such that* (1) *if* $xC_{z_i} = z_i x z_i^{-1}$ *for* $x \in G^*$, $f_i \beta^a C_{z_i} = \alpha f_i$, (2) $f_{i+1} C_{z_i} = f_i$, (3) $z_i \beta^a = z_{i+1}$, *and* (4) $X_{i+1} = X_i + a$. *For each element* $(g, x, y) \in e_i S e_i (i \in I \backslash N)$ *define* $(g, x, y)\theta = [z_i^{-1} \beta^{a(x-i-1)} \cdots z_i^{-1} \beta^a z_i^{-1} g f_i z_i \cdots z_i \beta^{a(y-i-1)}, X_i + a(x-i), X_i + a(y-i)]$ *if* $x > i$, $y > i$. *If* $x(y) = i$, *the left (right) multiplier of* gf_i *is* e^*, *the identity of* G^*. *The square brackets indicate an element of* S^*. *Then,* θ *is a homomorphism of S into* S^* *and conversely every homomorphism of S into* S^* *is obtained in this fashion.* $S \cong S^*$ *if and only if each* f_i *is an isomorphism of S onto* S^* *and* (1), (2), *and* (3) *are valid with* $a = 1$.

Proof. The proof involves an application of [**8**, Theorem 2.3, Theorem 1.1, and Theorem 1.2].

Every congruence ρ on an I-bisimple semigroup $S = (G, \alpha)$ is either a group congruence (S/ρ is a group) or an idempotent separating congruence (each ρ-class contains at most one idempotent). The group congruences are uniquely determined by the normal subgroups of the maximal group homomorphic image of S. ρ is idempotent separating if and only if $\rho = \rho^V((g, a, b)\rho^V(h, c, d)$ if and only if $a = c$, $b = d$, and $Vg = Vh$ where V is a subgroup of G such that $h(g\alpha^n)h^{-1} \in V$ for $h \in G$, $g \in V$, and $n \in I^0$). Results of [**4**] are significant here.

Theorem 3. *Let* $S = (G, \alpha)$ *and let e be the identity of G. If* $N = \{g \in G / g\alpha^n = e \text{ for some } n \in I^0\}$, *N is a normal subgroup of G. Let* $g \rightarrow \bar{g}$ *be the natural homomorphism of G onto G/N. If* $(xN)\theta = (x\alpha)N$, $x \in G$, θ *is an endomorphism of G/N. The maximal group homomorphic image H of S is isomorphic to* $G/N \times I$ *under the definition of equality,* $(\bar{g}, b-a) = (\bar{h}, d-c)$, $\bar{h}, \bar{g} \in G/N$, $a, b, c, d \in I^0$ *if there exist* $x, y \in I^0$ *such that* $x + b = y + d$, $x + a = y + c$, *and* $\bar{g}\theta^x = \bar{h}\theta^y$ *and the multiplication* $(\bar{g}, b-a)(\bar{h}, d-c) = (\bar{g}\theta^c \bar{h}\theta^b, (b+d)-(a+c))$. *The homomorphism of S onto H is given by* $(g, i+a, i+b)\theta = (\bar{g}, b-a)$ *where* $i \in I$, $a, b \in I^0$.

Proof. We utilize [**7**, pp. 431–434, especially Theorem 2.1]. q.e.d. If σ is the minimum group congruence on $S = (G, \alpha)$, $S/\mathcal{H} \cap \sigma \cong (G/N, \theta)$ (θ, N are defined in the statement of Theorem 3) and by [**3**] $S/\mathcal{H} \cup \sigma \cong (I, +)$.

To determine the (ideal) extensions of $S = (G, \alpha)$ by an arbitrary semigroup T, one utilizes the translational hull $\bar{S}$ of S [**2**, p. 140].

Theorem 4. *Let* $S = (G, \alpha)$ *and* $M = (I, G)$ *be the full group of mappings of I into G (pointwise multiplication).* $H = \{\beta \in M(I, G) / (i+1)\beta = (i\beta)\alpha \text{ for all } i \in I\}$ *is a subgroup of* $M(I, G)$. *Let* ρ_i $(i \in I)$ *be the inner* *right translation of* $(I, +)$ *determined by* i. *Thus,* $W = H \times I$ *under the multiplication* $(\beta, i)(\gamma, j) = (\beta \circ \rho_i\gamma, i+j)$ *where* $\circ$ *is the operation in* H *and juxtaposition denotes iteration of mappings is a group. Then* $\bar{S} = W \cup S$ *with multiplication* $(\beta, a)(g, i, j) = ((i-a)\beta \cdot g, i-a, j)$ *and* $(g, i, j)(\beta, a) = (g(j\beta), i, j+a)(S \cap W = \square)$.

COROLLARY. *Every extension of* $S = (G, \alpha)$ *by* $T = M^0(G^*; K, \Lambda; P) \cdot (T = M^0(G^*; K, K; \Delta))$ *is given by a partial homomorphism* [**15**, p. 522] *if* T *has proper divisors of zero.*

THEOREM 5. *Let* $S = (G, \delta)$ *and* $T = M^0(G^*; K, \Lambda, P)$. *Let the following functions be given*: $\psi: K \to I$, $\theta: \Lambda \to I$, $\alpha: K \to G$, $\beta: \Lambda \to G$, *and* γ *a homomorphism of* G^* *into* G *such that* $p_{\lambda i} \neq 0$ *implies* $\lambda\theta = i\psi$ *and* $(\lambda\beta)(i\alpha) = p_{\lambda i}\gamma$. *Then* ϕ *defined on* T^* *by* $*(a; i, \lambda)\phi = ((i\alpha)(a\gamma)(\lambda\beta); i\psi, \lambda\theta)$ *is a partial homomorphism of* T^* *into* S *and conversely every partial homomorphism of* T^* *into* S *is obtained in this fashion. If* $T = M^0(G^*; K, K; \Delta)$,* *becomes* $(a, i, j)\phi = ((i\alpha)(a\gamma)(j\alpha)^{-1}, i\psi, j\psi)$.

In the case $T^* = M(R; K, \Lambda; P)$ is completely simple one may give an explicit determination of the extensions of S by T in terms of a homomorphism of R into $(I, +)$, mappings of $R \to H$ (see statement of Theorem 4), $K \to H$, $K \to I$, $\Lambda \to H$, and $\Lambda \to I$ or by partial homomorphisms [**14**].

We next give a certain generalization of the bicyclic semigroup, C. Let $C \circ C$ denote $C \times C$ under the multiplication $((m, n), (k, t))((m', n'), (k', t')) = ((m, n)(m', n'), f(n, m'))$ where $f(n, m') = (k, t)$, $(k, t)(k', t')$, or (k', t') according to whether $n > m'$, $n = m'$, or $n < m'$. (See [**10**].) E_S is lexicographically ordered if and only if E_S is order isomorphic to $I^0 \times I^0$ under the order $(n, m) < (k, s)$ if $k < n$ or $k = n$ and $m > s$.

THEOREM 6. *Let* S *be a bisimple semigroup.* E_S *is lexicographically ordered if and only if* $\mathcal{H}$ *is a congruence on* S *and* $S/\mathcal{H} \cong C \circ C$. *If* S *has a trivial group of units* $S \cong C \circ C$.

PROOF. [**5**, Theorem 2.2] and [**8**, Theorem 1.2] are relevant.

The above definitions and theorems may be generalized to arbitrary finite dimensions. For a class of bisimple semigroups S such that E_S is lexicographically ordered, $S \cong G \times C \circ C$ where G is a certain group under a suitable multiplication [**11**].

Warne [**8**], [**11**] discussed the structure of bisimple inverse semigroups with identity on which $\mathcal{H}$ is a congruence. This is the case for all semigroups given here. However, let F be the positive part of any ordered field and let $P = (F\backslash 0) \times F$ under the multiplication

$(a, b)(c, d) = (ac, bc+d)$. If we substitute P in the Clifford construction [1], we obtain a bisimple inverse semigroup with identity on which $\mathcal{H}$ is not a congruence.

The results given here will appear in [11–14].

Added in proof. In [17], we give examples of bisimple inverse semigroups without identity on which $\mathcal{H}$ is not a congruence and the lexicographic case (with and without identity) is developed fully in [16] and [17].

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