ON MALLIAVIN'S COUNTEREXAMPLE TO SPECTRAL SYNTHESIS

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Malliavin's disproof of spectral synthesis breaks into two main parts: the first uses a certain "operational calculus," while the second involves a construction. Here we will be concerned only with the second part. The required construction is complicated, and several versions of it have been given [2], [3], [5]. In this note we describe an approach which appears to be somewhat simpler.

It should be mentioned that Varopoulos [6], [7] has recently given a completely different disproof of spectral synthesis, using tensor products of Banach algebras. However Malliavin's original counterexample, although difficult to construct, gives a very powerful result: for instance it shows that spectral synthesis fails even for principal ideals.

(Malliavin's results imply the existence of an $f \in A(\Gamma)$ such that $f, f^2, f^3, \cdots$ all generate different closed ideals in $A(\Gamma)$. The hypothesis of spectral synthesis asserts that any two closed ideals in $A(\Gamma)$ having the same "zero set" coincide—see below for definitions.)

DEFINITIONS. G is an infinite discrete abelian group; Γ is its dual, which is compact and not discrete. For $f \in L^1(G)$ the Fourier transform $\hat{f}$ is defined by $\hat{f}(\gamma) = \sum_G f(p)(-p, \gamma)dp$. For $f \in L^1(\Gamma)$ we set $\hat{f}(p) = \int_\Gamma f(\gamma)(p, \gamma)d\gamma$, so that the Fourier inversion theorem holds: $f^{\hat{}\hat{}} = f$. $A(\Gamma)$ denotes the algebra of Fourier transforms $\hat{f}(\gamma), f \in L^1(G)$. It is endowed with the norm, $\|\hat{f}\| \equiv \sum |f(p)|$, so that $A(\Gamma)$ is just an isomorphic and isometric copy of $L^1(G)$. The *zero set* of an ideal $I \subseteq A(\Gamma)$ is the set of points $\gamma_0 \in \Gamma$ such that $f(\gamma_0) = 0$ for all $f \in I$. For $g \in A(\Gamma)$, $\eta(g)$ denotes the L^∞ norm of the sequence $\hat{g}$: $\eta(g) \equiv \sup |\hat{g}(p)|$, $p \in G$.

Malliavin showed (cf. [3], [4], [5]) that spectral synthesis fails in $L^1(G)$ provided there exists a real valued function $f \in A(\Gamma)$ such that, for $-\infty < u < \infty$

$$\text{(A)} \qquad \eta(e^{iuf}) = O(|u|^{-n}), \quad \text{all } n \geqq 0.$$

THEOREM. *There exists a real valued function $f \in A(\Gamma)$ which satisfies* (A).

PROOF. To simplify the discussion, we will assume that G is the group of integers (Γ is the circle group). The modifications needed for dealing with the general case are similar to those in Rudin [5,

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pp. 177–181]. Actually (A) can be improved upon; in fact, for any $s>1$, there exists a real valued function $f\in A(\Gamma)$ such that

(B) $$\eta(e^{iuf}) = O[\exp(-\delta\,|\,u\,|^{1/s})] \qquad (\delta > 0).$$

This function has the form

(1) $$f(\gamma) = \sum_{k=1}^{\infty} (1/k^s) \cos p_k\gamma,$$

where $p_1<p_2< \cdots$ is a "very rapidly" increasing sequence of integers. For the sake of definiteness, let us take $s=2$.

The proof of (B) is based on certain properties of the norm η. These are given in (a), (b), (c), Lemma 1 and Lemma 2.

REMARKS. Although η is obviously a norm in the linear sense, it is not multiplicative (cf. Lemma 1). Proposition (c) depends on the fact that G is the group of integers, and Lemmas 1 and 2 would have to be modified slightly for the case of an arbitrary G. The proofs of (a)–(c) are trivial.

(a) If $h\in A(\Gamma)$, $|h|\leqq 1$, then $\eta(h)<1$ unless the Fourier series expansion of $h(\gamma)$ contains exactly one term.

(b) η is continuous in terms of the topology on $A(\Gamma)$ (since $\eta(\ )\leqq\|\ \|$).

(c) For all integers $k\neq 0$, $\eta[g(k\gamma)]$ is independent of $k(g\in A(\Gamma))$.

LEMMA 1. *Let g_α and h_α be nonzero elements of $A(\Gamma)$ which depend continuously on a real parameter α. Then for any $M>0$ and any $\epsilon>0$, there exists a positive integer k such that*

(2) $$\eta[g_\alpha(\gamma)\cdot h_\alpha(k\gamma)] < (1+\epsilon)\cdot\eta(g_\alpha)\cdot\eta(h_\alpha) \quad \textit{for all } \alpha\in[-M, M].$$

SKETCH OF PROOF. The idea is that, if k becomes very large, the nonzero Fourier coefficients of $h(k\gamma)$ will be very far apart. But by definition of $A(\Gamma)$, the Fourier series for $g_\alpha(\gamma)$ is absolutely convergent. Moreover, since g_α varies continuously in $A(\Gamma)$, the convergence can be shown to be uniform in α, $\alpha\in[-M, M]$. Then (2) follows from the standard convolution formula for the Fourier coefficients of a product.

LEMMA 2. *For any interval $[a, b]$ not containing 0, there exists a constant $\rho<1$ such that $\eta[\exp(i\alpha\cos k\gamma)]\leqq\rho$ for all $k\neq 0$ and all $\alpha\in[a, b]$.*

PROOF. Simply apply (a), (b), and (c) (a continuous function on a compact set assumes a maximum value).

Now to prove (B). Since $f(\gamma)=\sum_{k=1}^{\infty}(1/k^2)\cos p_k\gamma$, it follows that $\exp(iuf(\gamma))=\prod_{k=1}^{\infty} b_k(\gamma, u)$ where

(3) $$b_k(\gamma, u) \equiv \exp[(iu/k^2)\cos p_k\gamma].$$

Define $c_n(\gamma) \equiv \prod_{k=1}^{n} b_k(\gamma)$. Choose $\{\epsilon_k\}$, $\epsilon_k > 0$, so that $\prod(1+\epsilon_k) < 2$. Then from Lemma 1, *for a suitable choice of* $\{p_n\}$ (defined by induction of course)

$$\eta(c_{n+1}) < (1+\epsilon_n)\eta(c_n)\eta(b_{n+1}) \quad \text{for } |u| \leqq n^2. \tag{4}$$

(Note that the range of admissible values of u increases with each n.) Thus, since η is continuous on $A(\Gamma)$

$$\eta\left(\prod_{k=1}^{\infty} b_k\right) < 2\eta(c_n) \prod_{k>n} \eta(b_k) \quad \text{for } |u| \leqq n^2. \tag{5}$$

Now break the interval $[1, \infty)$ and the sequence $\{1, 2, \cdots\}$ into sections $4^{n-1} \leqq |u| \leqq 4^n$, $2^n < k \leqq 2^{n+1}$. By Lemma 2 there exists a $\rho < 1$ such that for any u, k with $1/16 \leqq |u/k^2| \leqq 1$, $\eta(b_k) \leqq \rho$ (cf. (3)). Hence (5) implies

$$\eta\left(\prod_{k=1}^{\infty} b_k\right) = O[\rho^{|u|^{1/2}}] \qquad (\rho < 1). \tag{B$'$}$$

Proof. Consider $|u|$ in the interval $4^{n-1} \leqq |u| \leqq 4^n$; now replace n by 2^n in (5), and look only at the terms $\eta(b_k)$ with $2^n < k \leqq 2^{n+1}$ ($\eta(b_i)$, $\eta(c_i) \leqq 1$ for all i, because $|b_i| \equiv |c_i| \equiv 1$).

Finally, since $\prod_{k=1}^{\infty} b_k = \exp(iuf)$, (B$'$) is equivalent to (B) (with $s = 2$).

Added in proof. The author has recently learned of another simplified proof of "non spectral synthesis" due to Kahane and Katznelson (Israel J. (1963); see also Stanford Univ. Seminar Notes by Katznelson, 1965). Their construction requires a little more analytic preparation than that given here; but it also proves more—e.g. that the function f described above can be chosen to be Hölder continuous.

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