# AN ASYMPTOTIC FORMULA FOR THE EIGENVALUES OF THE LAPLACIAN OPERATOR IN AN UNBOUNDED DOMAIN

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F. Rellich [5] and, more generally, A. M. Molcanov [3] have shown that the problem

$$(1)\qquad \begin{aligned} \tfrac{1}{2}\Delta^2 u(x) + \lambda u(x) &= 0, \qquad x \in \Omega \\ u(x) &= 0, \qquad x \in \partial\Omega \end{aligned}$$

has a discrete spectrum (and consequently a complete orthonormal system of eigenfunctions in $\mathfrak{L}_2(\Omega)$) provided that $\Omega$ is a "quasi-bounded" domain in $E_n$. A domain $\Omega$ is said to be quasi-bounded if it is either bounded or satisfies

$$\lim_{x\to\infty, x\in\Omega} \operatorname{dist}(x, \partial\Omega) = 0.$$

(See [1] for a proof of Molcanov's result, based on a generalization of the Kondrachoff embedding theorem for the Sobolev spaces $H_0^m(\Omega)$.) The problem of determining the asymptotic behavior of the eigenvalues of (1) has remained open (cf. [2, p. 233]).

In the present note we consider the above problem from the point of view of random processes, as described in detail for the case of a bounded domain, as well as for the case of the operator $-\frac{1}{2}\Delta^2 + V(x)$ (with $V(x) \to +\infty$ as $|x| \to \infty$) on an unbounded domain, in the papers of D. Ray [4] and M. Rosenblatt [6]. We will show that if $\Omega$ satisfies the following condition

$$(2)\qquad m(\Omega \cap [a < |x| < a + 1]) = O(a^{-\beta})$$

for some $\beta > \frac{1}{2}$, then simple modifications of Ray's arguments suffice to prove discreteness of the spectrum, as well as to obtain an asymptotic formula for the eigenvalues.

We take Ray's paper [4] as a starting point. Thus (assuming a cone condition for $\Omega$, as described in Theorem 1 below) we already have a Green's function $K(x, y, t)$ corresponding to the equation

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$$\Delta^2\phi(x,t) = 2\frac{\partial}{\partial t}\phi(x,t), \qquad x \in \Omega, \qquad t > 0, \tag{3}$$

and zero boundary conditions. We first wish to verify that the integral operator $K_t$ with kernel $K(x, y, t)$ is completely continuous in $\mathfrak{L}_2(\Omega)$. As in [4, Lemma 3] we see that it is sufficient to show, for fixed $t>0$,

$$\int_{\Omega\cap[|x|>a]} |K_t\psi(x)|^2\,dx \to 0 \quad \text{as } a \to \infty, \tag{4}$$

uniformly for $\psi \in \mathfrak{L}_2(\Omega)$, $\|\psi\| = 1$. But, as in [4],

$$\int_{\Omega\cap[|x|>a]} |K_t\psi(x)|^2\,dx$$
$$\leqq \int_{\Omega\cap[|x|>a]} dx\cdot\text{prob}\{x + x(\tau) \in \bar{\Omega},\ 0 \leqq \tau \leqq t\}\cdot\|\psi\|^2.$$

By an elementary calculation using (2), we have for any $\beta' < \beta$

$$\text{prob}\{x + x(\tau) \in \bar{\Omega},\ 0 \leqq \tau \leqq t\}$$
$$\leqq \text{prob}\{x + x(t) \in \bar{\Omega}\} = O(|x|^{-\beta'}), \qquad x \in \Omega;$$

here $t$ is fixed. Hence, writing $\Omega_i = \Omega \cap [i \leqq |x| < i+1]$, $i = 0, 1, 2, \cdots$, we have (taking $\beta' > \frac{1}{2}$)

$$\int_\Omega \text{prob}\{x + x(\tau) \in \bar{\Omega},\ 0 \leqq \tau \leqq t\}\,dx$$
$$= \sum_i \int_{\Omega_i} \text{prob}\{x + x(\tau) \in \bar{\Omega},\ 0 \leqq \tau \leqq t\}\,dx$$
$$= O\left(\sum_i i^{-2\beta'}\right) < \infty,$$

and (4) follows from this. We therefore have

THEOREM 1. *Let $\Omega$ be an open set in $E_n$, satisfying condition* (2) *and the following cone condition: for each $x \in \partial\Omega$ there is an open cone with vertex $x$, lying outside $\bar{\Omega}$. Let $K_t$ be the integral operator in $\mathfrak{L}_2(\Omega)$ with kernel $K(x, y, t)$.*

*Then $K_t$ is completely continuous and hence has a countable set of eigenvalues $\{\exp(-\lambda_j t),\ j = 0, 1, 2, \cdots\}$ with corresponding complete orthonormal eigenfunctions $\{\phi_j(x)\}$, which are independent of $t$. Moreover the $\lambda_j$ are eigenvalues and the $\phi_j$ eigenfunctions of the problem* (1).

COROLLARY. *Let $\Omega$ be as in Theorem* 1. *Then*

$$\sum_{\lambda_j<\lambda} \phi_j^2(x) \sim \left(\frac{\lambda}{2\pi}\right)^{n/2} \cdot \frac{1}{\Gamma(1 + n/2)}$$

*as $\lambda \to \infty$, for each $x \in \Omega$.*

The proofs of the asserted properties of the $\lambda_j$ and $\phi_j$ are the same as in Ray's paper. In particular, Ray shows that

$$\sum_j \exp(-\lambda_j t)\phi_j^2(x) = K(x, x, t) \sim \left(\frac{1}{2\pi t}\right)^{n/2} \tag{5}$$

as $t \to 0$, uniformly for $x \in \Omega$; in the present case this follows from the fact that $K(x, y, t)$ is a Hilbert-Schmidt kernel, as can be proved in a manner similar to the above verification of (4)—it is useful to notice that $0 \leqq K(x, y, t) \leqq (2\pi t)^{-n/2} \cdot \exp(-|x-y|^2/2t)$.

THEOREM 2. *Let $\Omega \subset E_n$ satisfy the hypotheses of Theorem* 1. *Let $\rho(x)$ be a nonnegative function in $\mathfrak{L}_1(\Omega)$. Define*

$$N_\rho(\lambda) = \sum_{\lambda_j \leqq \lambda} \int_\Omega \rho(x)\phi_j^2(x)\, dx.$$

*Then*

$$N_\rho(\lambda) \sim \left(\frac{\lambda}{2\pi}\right)^{n/2} \frac{1}{\Gamma(1 + n/2)} \int_\Omega \rho(x)\, dx. \tag{6}$$

PROOF. Applying (5) to the Laplace-Stieltjes transform of $N_\rho(\lambda)$, we have

$$\begin{aligned} \int_0^\infty e^{-\lambda t}\, dN_\rho(\lambda) &= \int_\Omega \rho(x) \sum_j \exp(-\lambda_j t)\phi_j^2(x)\, dx \\ &= \int_\Omega \rho(x) K(x, x, t)\, dx \\ &\sim \left(\frac{1}{2\pi t}\right)^{n/2} \int_\Omega \rho(x)\, dx. \end{aligned}$$

Hence the Tauberian theorem of Karamata applies, and yields (6). q.e.d.

We obviously obtain the classical formula of Weyl if we put $\rho(x) \equiv 1$ on a bounded region (or even on an unbounded region of finite volume). If in the general case we choose a bounded function $\rho(x)$, we obtain $N_\rho(\lambda) \leqq c \cdot N(\lambda)$ where $N(\lambda) = N_1(\lambda)$ is the usual function; we therefore obtain a one-sided estimate for $N(\lambda)$:

$$N(\lambda) \succ \lambda^{n/2}$$

where $f(\lambda) \succ g(\lambda)$ means the same as $g(\lambda) = O(f(\lambda))$. We remark that our results are unaffected if the operator $-\Delta^2$ is replaced by $-\Delta^2 + V(x)$ if $V(x)$ is a bounded function on $\Omega$.

The foregoing results can also be derived using analytical methods similar to those of Titchmarsh [7]; the basic properties of the Green's function $G(x, y, \lambda)$ in this case are due to D. Hewgill (Thesis, University of British Columbia).

## References


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