RATIONAL POINTS IN HENSELIAN DISCRETE VALUATION RINGS

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Let R be a Henselian discrete valuation ring (i.e., one in which Hensel's Lemma holds; examples are complete discrete valuation rings, the ring of algebraic p-adic integers, the ring of convergent power series in one variable over a complete valued field [1]). Let t be a generator of the maximal ideal, K the field of fractions, R^* the completion of R; let K^* be the field of fractions of R^*. If $F = (F_1, \cdots, F_r)$ is a system of r polynomials in n variables with coefficients in R, let $FR[X]$ be the ideal in $R[X]$ generated by $F_1, \cdots, F_r$. If x is an n-tuple with coordinates in R, set $F(x) = (F_1(x), \cdots, F_r(x))$:

THEOREM 1. *Assume that K^* is separable over K. Then there are integers $N \geqq 1$, $c \geqq 1$ depending on $FR[X]$ such that for any $\nu \geqq N$ and any x in R such that*

$$F(x) \equiv 0 \pmod{t^\nu}$$

there exists y in R such that

$$y \equiv x \pmod{t^{[\nu/c]}},$$

$$F(y) = 0.$$

COROLLARY 1. *Let Y be a prescheme of finite type over R. Then there are integers $N \geqq 1$, $c \geqq 1$ depending on Y such that for $\nu \geqq N$ and for any point x of Y in R/t^ν, the image of x* mod $t^{[\nu/c]}$ *lifts to a point of Y in R.*

COROLLARY 2. *Y has a point in R if and only if Y has a point in R/t^ν for all ν.*

Corollary 1 follows from Theorem 1 by taking a finite covering of Y by affine opens Y_i and remarking that $Y(S) = \bigcup_i Y_i(S)$ for any local R-algebra S.

Let $Y = \operatorname{Spec} R[X]/FR[X]$ be the affine scheme over R defined by F, Y_K the scheme over K obtained by base change. In the special case that R is complete and Y_K is irreducible and smooth over K, Néron [2; Proposition 20, p. 38] has proved a different form of Theorem 1.

The proof of Theorem 1 goes by Noetherian induction on Y_K. One reduces easily to the case Y reduced and irreducible. Then there are two cases, depending on whether the function field of Y_K is separable or not over K. In the separable case, the key is Newton's Lemma, which enables us to refine x to a zero provided that Y is a complete intersection and the Jacobian matrix of F at x has the maximal rank mod $t^{[(\nu-1)/2]}$; if the latter condition fails, then the inductive hypothesis enables us to refine x to a zero on the singular locus of Y_K. In the inseparable case, there is a finite purely inseparable extension K' of K such that $Y_{K'}$ is not reduced. Since K^* is separable over K, the integral closure R' of R in K' is a finite R-module [**3**; 0_{IV}, 23.1.7(ii)]. Then techniques of [**4**] enable us to pull $(Y_{R'})_{\text{red}}$ down to a proper closed subscheme of Y for which the inductive hypothesis applies.

The detailed proof will appear in Publ. Math. Inst. Hautes Études.

As one application of Theorem 1, recall that a domain R is called C_i if any form with coefficients in R of degree d in n variables with $n > d^i$ has a nontrivial zero in R.

THEOREM 2. *If k is a C_i field, then the field $k((t))$ of formal power series in one variable t over k is C_{i+1}.*

This generalizes some results of Lang [**5**], who did the cases $i=0$ and k finite.

It suffices to prove that $R=k[[t]]$ is C_{i+1}. By Lang [**5**], $k[t]$ is C_{i+1}. Hence the hypersurface H in projective $(n-1)$-space defined by the given form has a point in the ring R/t^ν for all ν. By Corollary 2, H has a point in R.

Note. The same type of argument yields a short proof of Lang's theorem that if R is a Henselian discrete valuation ring with algebraically closed residue field, such that K^* is separable over K, then R is C_1. For by Corollary 2, we may assume R complete, and since C_1 is inherited by finite extensions, we may also assume R unramified. Then the argument given in [**5**; p. 384] shows that H has a point in R/t^ν for all ν.

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