ON POINCARÉ'S BOUNDS FOR HIGHER EIGENVALUES

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1. **Introduction.** Let A be a compact symmetric negative-definite operator on a real Hilbert space H having the inner product (u, v). Let $\lambda_1 \leqq \lambda_2 \leqq \cdots$ be the eigenvalues and $u_1, u_2, \cdots$ the corresponding orthonormal set of eigenvectors of the equation $Au = \lambda u$. Denote by $R(u)$ the Rayleigh quotient $(Au, u)/(u, u)$. For a given λ_n let m and N be the smallest and largest indices respectively such that $\lambda_m = \lambda_n = \lambda_N$. There are two variational characterizations of λ_n by inequalities. One goes back to Poincaré [1, p. 259] and was reformulated by Pólya and Schiffer [2], [3]. The other is the maximum-minimum principle for which A. Weinstein [4], [5] recently introduced a new approach. Using the Weinstein determinant and the corresponding quadratic form he gave for the first time a complete discussion of the corresponding inequalities including the necessary and sufficient conditions for equality. In the present paper we give a similar discussion of Poincaré's characterization of λ_n.

2. **The main result.** Let V_r be any r-dimensional subspace of H and let $p_1, p_2, \cdots, p_r$ be a basis for V_r. We consider the determinant

$$\det\{(Ap_i, p_k) - \lambda(p_i, p_k)\}, \qquad i, k = 1, 2, \cdots, r. \tag{1}$$

Using Parseval's formula we see that (1) can also be written as

$$\det\left\{\sum_{j=1}^{\infty} (\lambda_j - \lambda)(p_i, u_j)(p_k, u_j)\right\}, \qquad i, k = 1, 2, \cdots, r. \tag{2}$$

Let us note in passing the remarkable, but until now unexplained, similarity between (2) and the Weinstein determinant

$$W(\lambda) = \det\left\{\sum_{j=1}^{\infty} (\lambda_j - \lambda)^{-1}(p_i, u_j)(p_k, u_j)\right\}, \quad i, k = 1, 2, \cdots, r. \tag{3}$$

We can now formulate our main result.

THEOREM. *For any choice of V_r we have the inequality*

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$$\lambda_n \leqq \max_{u \in V_r} R(u) \tag{4}$$

if and only if $m \leqq r$. *By varying* V_r *we obtain the following characterization of* λ_n.

$$\lambda_n = \operatorname*{Min}_{V_r} \max_{u \in V_r} R(u), \qquad m \leqq r \leqq N. \tag{5}$$

Assuming that $m \leqq r$, *the necessary and sufficient conditions on the space* V_r *for the equality*

$$\lambda_n = \max_{u \in V_r} R(u) \tag{6}$$

are that $r \leqq N$ *and for any* $\epsilon > 0$ *the quadratic form with the symmetric matrix*

$$\{(Ap_i, p_k) - (\lambda_n + \epsilon)(p_i, p_k)\}, \quad i, k = 1, 2, \cdots, r \tag{7}$$

is negative definite.

Proof. The proofs of (4) and (5) have been given in [1] and [2], [3] for the case $r = n$. Obviously (4) holds also for $m \leqq r$ since $\lambda_m = \lambda_n$. To show the necessity of this condition we assume for the moment that (4) holds for all V_r where $r < m$ and choose V_r to be the subspace spanned by $u_1, u_2, \cdots, u_r$. In this case we have

$$\max_{u \in V_r} R(u) = \lambda_r < \lambda_m = \lambda_n \leqq \max_{u \in V_r} R(u)$$

which is a contradiction. As in [2], [3] the equality (5) follows immediately not only for $r = n$ but also for $m \leqq r \leqq N$. In fact, it is sufficient to use the classical choice $p_k = u_k$, $k = 1, 2, \cdots, r$ in order to obtain (6). In §3 we give an example which shows that the classical choice is not a necessary condition for (6). To prove our necessary and sufficient conditions we shall assume that the basis $p_1, p_2, \cdots, p_r$ has been chosen so that the matrix (7) is diagonal. First we show that our conditions are necessary. Suppose that (6) holds for $r > N$. Then, using (4), we obtain the contradiction

$$\lambda_r \leqq \max_{u \in V_r} R(u) = \lambda_n = \lambda_N < \lambda_r.$$

Since (6) implies

$$R(p_i) = (Ap_i, p_i)/(p_i, p_i) < \lambda_n + \epsilon, \; i = 1, 2, \cdots, r$$

all elements on the diagonal of (7) are negative, which proves that the quadratic form corresponding to (7) must be negative definite. To prove sufficiency we assume that for any $\epsilon > 0$ the diagonal matrix (7) is negative definite so that

$$(8) \qquad (Ap_i, p_i) < (\lambda_n + \epsilon)(p_i, p_i), \; i = 1, 2, \cdots, r$$

and

$$(9) \qquad (Ap_i, p_k) = (\lambda_n + \epsilon)(p_i, p_k), \; i \neq k; \; i, k = 1, 2, \cdots, r.$$

Since every $u \in V_r$ can be written as $u = \sum_{i=1}^{r} \gamma_i p_i$ we have

$$(10) \qquad R(u) = \frac{\sum_{i=1}^{r} \gamma_i^2 (Ap_i, p_i) + \sum_{i \neq k} \gamma_i \gamma_k (Ap_i, p_k)}{\sum_{i,k=1}^{r} \gamma_i \gamma_k (p_i, p_k)}.$$

Using (8) and (9) in (10) we get for every $u \in V_r$ $R(u) < \lambda_n + \epsilon$. Combining this with (4) we have $\lambda_n \leqq \max_{u \in V_r} R(u) \leqq \lambda_n + \epsilon$. Since ϵ can be chosen arbitrarily small the equality (6) holds.

3. **Example.** We now give an example in which (6) holds for a nonclassical choice of V_r. Let $\lambda_1 < \lambda_2 < \lambda_3$ and let $m = r = n = N = 2$. We choose $p_1 = u_2$ and $p_2 = u_1 + \beta u_3$ as a basis for V_2 where $0 < \beta^2 \leqq (\lambda_2 - \lambda_1)/(\lambda_3 - \lambda_2)$. A simple calculation shows that for every $u \in V_2$ the inequality $R(u) \leqq \lambda_2$ is satisfied. Since $R(u_2) = \lambda_2$ we have $\lambda_2 = \max_{u \in V_2} R(u)$. In this case (7) is a diagonal matrix with elements $-\epsilon$, $-\epsilon(1+\beta^2)$, which verifies our criterion. Let us note the formal analogy to the new maximum-minimum theory of A. Weinstein, where the quantities $(\lambda_j - \lambda)^{-1}$, $\lambda_n - \epsilon$, and β^{-1} appear in place of $\lambda_j - \lambda$, $\lambda_n + \epsilon$, and β.

4. **Concluding remark.** It has been shown in [1] and [2], [3] that the roots $\lambda_1' \leqq \lambda_2' \leqq \cdots \leqq \lambda_r'$ of (1) satisfy the inequalities

$$\lambda_1 \leqq \lambda_1', \; \lambda_2 \leqq \lambda_2', \cdots, \lambda_r \leqq \lambda_r'$$

and that the simultaneous equalities

$$(11) \qquad \lambda_1 = \lambda_1', \; \lambda_2 = \lambda_2', \cdots, \lambda_r = \lambda_r'$$

are obtained by choosing $p_k = u_k$, $k = 1, 2, \cdots, r$. In another paper we shall prove that the only V_r for which (11) holds are those subspaces generated by eigenvectors belonging to $\lambda_1, \lambda_2, \cdots, \lambda_r$.

References

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