AHLFORS' CONJECTURE CONCERNING EXTREME SARIO OPERATORS

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The linear operators, introduced by Sario [4] to construct harmonic functions with prescribed properties on Riemann surfaces, form a convex set. Ahlfors [1] has conjectured a representation for the extreme operators of this convex set. We give an equivalent formulation of this conjecture and show that it is not true in general.

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1. Let W be a subregion of a Riemann surface R. We suppose W has a compact complement and that its relative boundary α is analytic. We consider a linear operator T which, to continuous values f on α, assigns a harmonic function Tf on W such that $Tf=f$ on α. T is assumed to have the following additional properties:

$$T1 = 1, \qquad Tf \geqq 0 \quad \text{if } f \geqq 0, \tag{1.1}$$

$$\int_\alpha \frac{\partial Tf}{\partial n} ds = 0. \tag{1.2}$$

Sario [4] has called these operators *normal linear operators*. It is clear that the set of such operators on W form a convex set.

2. We assume, with Ahlfors [1], that the ideal boundary β of R is analytic. Consider the harmonic measure of the region befween α and β. That is, the harmonic function on W which is 0 on α and 1 on β and normalized so that the period of its conjugate function along α is 1. In terms of this conjufate function we parametrize α and β by $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, respectively.

Given f on α, Tf has radial limits almost everywhere on β and this null set E may be selected independent of f (See [1]). In this manner we may consider T as inducing a linear mapping from the space $C(0, 1)$ of continuous functions on α to $L^\infty(0, 1)$ the space of bounded measurable functions on β. We denote this induced linear operator by T also and the class of all such operators by L. They have the following properties corresponding to conditions (1.1) and (1.2) above:

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$$T1 = 1, \qquad Tf \geqq 0 \quad \text{if } f \geqq 0, \tag{2.1}$$

$$\int_0^1 Tf = \int_0^1 f. \tag{2.2}$$

For a fixed value of y the mapping $f \to (Tf)(y)$ defines a linear functional on $C(0, 1)$. By the Riesz representation theorem we have

$$(Tf)(y) = \int_0^1 f(x)\, d\mu(x, y). \tag{2.3}$$

Corresponding to the above conditions on T, the family of measures $\{\mu(\cdot, y)\}$ on $0 \leqq x \leqq 1$ associated with T has the following properties:

$$\mu([0, 1], y) = 1, \qquad \mu(\cdot, y) \geqq 0 \tag{2.4}$$

$$\int_0^1 dy \int_0^1 f(x)\, d\mu(x, y) = \int_0^1 f(x)\, dx. \tag{2.5}$$

3. Ahlfors [1] has conjectured that T is extreme in L if and only if $\mu(\cdot, y)$ is a point mass for almost every y. If this were the case, it is clear that the corresponding operator T is extreme. We note, in this case, that we have a mapping $x = g(y)$ defined for almost every y which assigns to y the point $x = g(y)$ where the point mass is concentrated. Hence $(Tf)(y) = f(g(y))$ and condition (2.5) becomes

$$\int_0^1 f(g(y))\, dy = \int_0^1 f(x)\, dx. \tag{3.1}$$

Since this holds for every f in $C(0, 1)$, $g(y)$ is a measure preserving transformation from $0 \leqq y \leqq 1$ to $0 \leqq x \leqq 1$.

Conversely, if $g(y)$ is any measure preserving transformation from $0 \leqq y \leqq 1$ to $0 \leqq x \leqq 1$ then the composition operator defined by $(Tf)(y) = f(g(y))$ is extreme in L and the associated mass distribution $\mu(\cdot, y)$ is a point mass for almost every y. Thus an equivalent formulation of Ahlfors' conjecture is that every extreme operator of L is of the form $(Tf)(y) = f(g(y))$ where $g(y)$ is a measure preserving transformation from $0 \leqq y \leqq 1$ to $0 \leqq x \leqq 1$.

4. Consider the class K of linear operators which map $C(0, 1)$ into $L^\infty(0, 1)$ and satisfy the conditions:

$$T1 = 1, \qquad Tf \geqq 0 \quad \text{if } f \geqq 0. \tag{4.1}$$

Then $L = \{T \in K : \int Tf = \int f,\ f \in C(0, 1)\}$.

K forms a convex set and we determine the extreme operators of K.

Our method is to replace the space $L^\infty(0, 1)$ by an equivalent $C(S)$ space where S is compact Hausdorff and apply the results of Phelps [2] concerning extreme operators which map $C(0, 1)$ into $C(S)$.

The Gel'fand representation (See Dunford and Schwartz, vol. 1, p. 445) gives us an isometric isomorphism Λ between $L^\infty(0, 1)$ and $C(S)$ where S is a compact Hausdorff space. The isomorphism Λ maps positive functions into positive functions and is an algebraic isomorphism. That is, if $h(y)=f(y)g(y)$ almost everywhere then $\Lambda h=\Lambda f\Lambda g$. Also if F is an arbitrary continuous function of a complex variable and h is in $L^\infty(0, 1)$ then

$$\Lambda(F(h)) = F(\Lambda(h)). \tag{4.2}$$

Note that under the mapping Λ a positive function has a positive function as a preimage. For if $\Lambda(f)=g$ where $g\geqq 0$ then taking F in (4.2) to be the absolute value function we have $\Lambda(|f|)=|\Lambda(f)|=|g| =\Lambda(f)$. Therefore $f=|f|$ or $f\geqq 0$.

Set $T'f=\Lambda(Tf)$. Then T' is a linear mapping from $C(0, 1)$ to $C(S)$. Denote by K' the image of the class K under the map Λ, then the operators T' in K' satisfy:

$$T'1 = 1, \qquad T'f \geqq 0 \quad \text{if } f \geqq 0. \tag{4.3}$$

Moreover, T is extreme in K if and only if T' is extreme in K'.

THEOREM. *T is extreme in K if and only if there is a bounded measurable mapping g from $0\leqq y\leqq 1$ to $0\leqq x\leqq 1$ such that $Tf=f(g)$.*

PROOF. By Phelps theorem [2], T' in K' is extreme if and only if there is a continuous mapping G from S to $0\leqq x\leqq 1$ such that $T'f =f(G)$ for all f in $C(0, 1)$. But $\Lambda(Tf)=T'f=f(G)$ and since G is in $C(S)$ let g be in $L^\infty(0, 1)$ such that $\Lambda g=G$. Note that $g\geqq 0$ since $G\geqq 0$. Also since Λ is norm preserving we have $\|g\|=\|G\|\leqq 1$. Hence $0\leqq g\leqq 1$ and thus g is a bounded measurable mapping from $0\leqq y\leqq 1$ to $0\leqq x\leqq 1$. From (4.2) we have $\Lambda(f(g))=f(\Lambda(g))$ for every f in $C(0, 1)$ hence $\Lambda(Tf)=f(G)=f(\Lambda(g))=\Lambda(f(g))$ or $Tf=f(g)$.

REMARK. Phelps [2] has also shown that the extreme operators of K' can be characterized as the multiplicative operators. That is, $T'(fh)=(T'f)(T'h)$. Hence from $\Lambda(TfTh)=\Lambda(Tf)\Lambda(Th)=(T'f)(T'h) =T'(fh)=\Lambda(T(fh))$ we conclude that

THEOREM. *T is extreme in K if and only if T is multiplicative. That is, $T(fh)=TfTh$ for every f, h in $C(0, 1)$.*

5. We now show the connection between Ahlfors' conjecture and

the extreme operators of L and K. As noted earlier, Ahfors' conjecture may be stated as:

T is extreme in L if and only if $Tf=f(g)$ where g is a measure preserving mapping from $0\leqq y\leqq 1$ to $0\leqq x\leqq 1$.

THEOREM. *Ahlfors' conjecture holds if and only if every T which is extreme in L is also extreme in K.*

PROOF. If the conjecture holds then it is clear that T is multiplicative and hence extreme in K. Conversely if T extreme in L implies T extreme in K then $Tf=f(g)$ and the integrability condition (2.2) implies that g is measure preserving.

An equivalent formulation of the above result is

THEOREM. *Ahlfors' conjecture holds if and only if every extreme operator T in L is multiplicative.*

6. An example due to Ryff [**3**] shows that there exists an operator T in L which is extreme but not multiplicative. This example settles Ahlfors' conjecture in the negative.

Define the operator T as follows.

$$(Tf)(y) = [f(y/2) + f((y + 1)/2)]/2. \tag{6.1}$$

Simple calculations show that T is in L and is not multiplicative. Note that

$$(Tf)(y) = \int_0^1 f(x)\, d\omega(x, y) = [f(y/2) + f((y + 1)/2)]/2 \tag{6.2}$$

implies that the associated measure $\omega(\cdot, y)$ consists of a $1/2$ unit point mass at $y/2$ and a $1/2$ unit point mass at $(y+1)/2$.

To show that T is extreme in L, we show that if T can be represented as $T=(T_1+T_2)/2$ with T_1, T_2 in L, then $T=T_1=T_2$. In terms of the associated measures we have

$$\omega(\cdot, y) = [\mu(\cdot, y) + \nu(\cdot, y)]/2 \tag{6.3}$$

where μ and ν are the measures corresponding to T_1 and T_2, respectively.

Since μ and ν are nonnegative it is evident tlat each has mass concentrated only at the points $y/2$ and $(y+1)/2$ for almost every y. Also, from the fact that $\omega((y+1)/2, y)=\omega(y/2, y)=1/2$ for a.e. y we have

$$1 = \mu(y/2, y) + \nu(y/2, y) \tag{6.4}$$

and

$$(6.5)\qquad 1 = \mu((y+1)/2, y) + \nu((y+1)/2, y).$$

Let $\alpha(y)=\mu(y/2, y)$ and $\beta(y)=\mu((y+1)/2, y)$ then since the total mass of μ is 1 we have $\beta(y)=1-\alpha(y)$ for a.e. y. We show that $\alpha(y)=1/2$ for a.e. y and hence that $\mu(\cdot, y)=\nu(\cdot, y)$ and thus $T=T_1=T_2$.

From the nature of the mass distribution $\mu(\cdot, y)$ we have

$$(6.6)\qquad \int_0^1 f(x)\,d\mu(x, y) = \alpha(y)f(y/2) + [1-\alpha(y)]f((y+1)/2)$$

for every f in $C(0, 1)$. From the integrability condition (2.5) we obtain

$$(6.7)\qquad \int_0^1 f(x)\,dx = \int_0^1 \{\alpha(y)f(y/2) + [1-\alpha(y)]f((y+1)/2)\}\,dy.$$

Let f in $C(0, 1)$ be such that f vanishes on $[1/2, 1]$ then (6.7) becomes

$$(6.8)\qquad \int_0^1 f(x)\,dx = \int_0^1 \alpha(y)f(y/2)\,dy$$

or making a simple change of variable (6.8) is equivalent to

$$(6.9)\qquad \int_0^{1/2} [2\alpha(2t) - 1]f(t)\,dt = 0.$$

This holds for every continuous f on $[0, 1/2]$ which vanishes at the endpoints. Hence $\alpha(t)=1/2$ for a.e. t on $[0, 1]$.

References

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