EXISTENCE THEORY FOR TWO POINT BOUNDARY VALUE PROBLEMS[1]

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We consider second order two point boundary value problems of the form:

$$y'' = f(t, y, y'), \quad a \leqq t \leqq b; \tag{1}$$

$$a_0 y(a) - a_1 y'(a) = \alpha, \quad |a_0| + |a_1| \neq 0; \tag{2}$$

$$b_0 y(b) + b_1 y'(b) = \beta, \quad |b_0| + |b_1| \neq 0. \tag{3}$$

Our basic result is the

THEOREM. *Let $f(t, u, u')$ have continuous derivatives which satisfy:*

$$\frac{\partial f(t, u(t), u'(t))}{\partial u} > 0, \quad \left| \frac{\partial f(t, u(t), u'(t))}{\partial u'} \right| \leqq M, \tag{4}$$

for some $M \geqq 0$, $a \leqq t \leqq b$ and all continuously differentiable functions $u(t)$. Let the constants a_i, b_i satisfy:

$$a_i \geqq 0, \quad b_i \geqq 0, \quad i = 0, 1; \quad a_0 + b_0 > 0. \tag{5}$$

Then a unique solution of (1), (2), (3) *exists for each* (α, β).

PROOF. We sketch the proof. The initial value problem

$$\begin{aligned} &u'' = f(t, u, u'), \quad a \leqq t \leqq b; \\ &\left.\begin{aligned} a_0 u(a) - a_1 u'(a) &= \alpha \\ c_0 u(a) - c_1 u'(a) &= s \end{aligned}\right\}, \quad a_1 c_0 - a_0 c_1 = 1; \end{aligned} \tag{6}$$

has the unique solution $u(s; t)$. The problem (1), (2), (3) has as many solutions as there are real roots, s^* of

$$\phi(s) \equiv b_0 u(s; b) + b_1 u'(s; b) = \beta.$$

Since $u(s; t)$ is continuously differentiable with respect to s the derivative $\xi(t) \equiv \partial u(s; t)/\partial s$ satisfies the variational problem [1],

$$\xi'' = p(t)\xi' + q(t)\xi,$$

$$\xi(a) = a_1, \quad \xi'(a) = a_0;$$

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where the parameter s has been suppressed and

$$p(t) \equiv \frac{\partial f(t, u(s, t), u'(s, t))}{\partial u'}, \qquad q(t) \equiv \frac{\partial f(t, u(s, t), u'(s, t))}{\partial u}.$$

By a maximum principle argument, using $q(t)>0$, it can be shown that $\xi(t)$ is positive in $a<t\leqq b$. Then the differential inequality

$$\xi'' \geqq p(t)\xi'$$

follows and from it we deduce, with the aid of the positive integrating factor $\exp -\left[\int_a^t p(\tau)d\tau\right]$, that

$$\xi(t) > a_1 + a_0\left[\frac{1-\exp(-M(t-a))}{M}\right] > 0,$$

$$\xi'(t) > a_0 \exp(-M(t-a)) \geqq 0.$$

But then, by (5),

$$\frac{d\phi(s)}{ds} = b_0\xi(b) + b_1\xi'(b) \tag{7}$$

is positive and is bounded away from zero. Thus $\phi(s)=\beta$ always has one and only one root and the proof is concluded.

It is easy to weaken the hypothesis of the theorem in various ways and to extend the result to more general boundary conditions. For instance it need only be required that $\int_a^t p(\tau)d\tau$ be bounded for $a\leqq t\leqq b$ and all solutions $u(s, t)$ of (6). The condition $q(t)>0$ can be relaxed to allow equality in special cases. One of the boundary conditions, say (3), can be replaced by

$$b_0y(b) + b_1y'(b) + b_2y(a) + b_3y'(a) = \beta, \quad |b_0| + |b_1| \neq 0 \tag{8}$$

if in addition to (5) we require

$$b_2a_0 + b_3a_1 \geqq 0. \tag{9}$$

Nonlinear boundary conditions of the forms

$$g_1(y(a), -y'(a)) = 0, \tag{10}$$

$$g_2(y(a), y'(a), y(b), y'(b)) = 0, \tag{11}$$

can replace (2) and (8), respectively, if the functions $g_1(x_1, x_2)$ and $g_2(x_1, x_2, x_3, x_4)$ have continuous partial derivatives $g_{i,j}\equiv\partial g_i/\partial x_j$ which satisfy in place of (5) and (9):

$$(12) \qquad g_{i,j} \geqq 0 \begin{cases} i=1, \ j=1, 2 \\ i=2, \ j=3, 4 \end{cases}; \quad \begin{matrix} g_{1,1}+g_{1,2} \geqq \epsilon > 0 \\ g_{2,3}+g_{2,4} \geqq \epsilon > 0; \\ g_{1,1}+g_{2,3} > 0 \end{matrix} \quad g_{2,1}g_{1,2}+g_{2,2}g_{1,1} \geqq 0.$$

The implicit function theorem can be employed under this hypothesis and so we are assured that the appropriate initial value problems which replace (6) can be solved uniquely.

By strengthening condition (4) to require that for some positive constant N,

$$0 < \frac{\partial f}{\partial u} \leqq N,$$

the proof of the theorem becomes essentially constructive. That is we now show that

$$0 < \gamma \leqq \frac{d\phi(s)}{ds} \leqq \Gamma$$

where the constants γ and Γ are specific functions of N, M, $L \equiv b-a$, and the coefficients a_i, b_i. But then for any fixed m in

$$0 < m < 2/\Gamma$$

it follows by the contractive mapping principle that the sequence $\{s_\nu\}$ defined by

$$(13) \qquad s_{\nu+1} = s_\nu - m[\phi(s_\nu) - \beta], \qquad \nu = 0, 1, 2, \cdots$$

with s_0 arbitrary, converges to the unique root of $\phi(s)=\beta$. Since constructive existence proofs for the initial value problem (6) are known [1], the above yields such a proof for boundary value problems of the form (1), (2), (3). By imposing upper bounds on the derivatives in (12) we can also obtain a constructive existence proof for the problem (1), (10), (11).

It is clear that with little additional effort we can now prove the convergence of various numerical methods for solving boundary value problems of the indicated forms. In particular the shooting method is suggested and in practice one would employ some higher order iteration scheme, say Newton's method, in place of (13). A study of many such numerical methods as well as details of the proofs of some of the above results will be found in [2].

Recent results in [3] yield existence and uniqueness for (1) subject to more restrictive boundary conditions than (2), (3) but under much weaker conditions than (4).

References

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. H. B. Keller, *Numerical methods for two point boundary value problems*, (to appear).

3. P. Bailey, L. F. Shampine and P. Waltman, *Existence and uniqueness of solutions of the second order boundary value problem*, Bull. Amer. Math. Soc. **72** (1966), 96–98.

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