SOME RESULTS ON THE STABLE HOMOTOPY GROUPS OF SPHERES

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In this paper, we examine a particular example of a long-known spectral sequence in order to answer the lowest dimensional unresolved question about the stable homotopy groups of spheres. In particular we completely determine the p-primary component of $\pi_n(S)$ for $n \leqq 2(p-1)(p^2+2p)-6$, for odd primes p.

There is a spectral sequence [7] $\{E^r, d^r\}$ such that $E^2_{**} = H_*(X; \pi_*(S))$ and E^∞_{**} is a bigraded group associated with the stable homotopy groups of X, any topological space. This spectral sequence may be generalized by replacing space X by spectrum $\boldsymbol{A}$: $E^2_{**} = H_*(\boldsymbol{A}; \pi_*(S))$ and E^∞_{**} is a bigraded group associated with $\pi_*(\boldsymbol{A})$.

Consider this spectral sequence for $\boldsymbol{A} = \boldsymbol{K}(Z)$, the Eilenberg-MacLane spectrum $(A_n = K(Z, n))$. Then $E^\infty_{0,0} = Z$ and $E^\infty_{s,t} = 0$ for $(s, t) \neq (0, 0)$, because $\pi_*(\boldsymbol{K}(Z)) = \pi_0(\boldsymbol{K}(Z)) = Z$.

$E^2_{**} = H_*(\boldsymbol{K}(Z); \pi_*(S))$ is a tensor and torsion product of a well-known ring, $H_*(\boldsymbol{K}(Z))$ [1], and a ring about which information is sought, $\pi_*(S)$. This relation gives us the information needed.

For the actual computations, we replace S by the spectrum $\boldsymbol{L}_p$ where $(\boldsymbol{L}_p)_n = M(Z_p, n)$ is a Moore space of homology type (Z_p, n). It is easily shown that $\pi_n(\boldsymbol{L}_p) \cong \pi_n(S) \otimes Z_p + \mathrm{Tor}(\pi_{n-1}(S), Z_p)$, p odd. Thus given any $\theta \epsilon \pi_n(S)$ not divisible by p we have an element also called $\theta \epsilon \pi_n(\boldsymbol{L}_p)$. Given any $\eta \epsilon \pi_n(S)$ of order p, we have an element $\eta' \epsilon \pi_{n+1}(\boldsymbol{L}_p)$. ($\eta'$ can be constructed using Toda's toral construction [6] and is defined up to indeterminacy $\pi_{n+1}(S) \otimes Z_p \subset \pi_{n+1}(\boldsymbol{L}_p)$.)

Multiplication can be defined making $\pi_*(\boldsymbol{L}_p)$ an algebra over Z_p. Then $E^\infty_{**} = E^\infty_{0,0} = Z_p$ and

$$E^2_{**} = H_*(\boldsymbol{K}(Z); \pi_*(\boldsymbol{L}_p)) = H_*(\boldsymbol{K}(Z); Z_p) \otimes \pi_*(\boldsymbol{L}_p) = A_* \otimes \pi_*(\boldsymbol{L}_p)$$

where A_* is isomorphic to $E(\tau_1, \tau_2, \cdots) \otimes P(\xi_1, \xi_2, \cdots)$, the exterior algebra on generators τ_i tensored with the polynomial algebra on generators ξ_i, where $\dim \tau_i - 1 = \dim \xi_i = 2(p^i - 1)$. A_* is the dual algebra to the quotient coalgebra of the Steenrod Algebra, by the left ideal generalized by the Bockstein.

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Toda [5] had found elements $\alpha_n \epsilon {}_p\pi_{2n(p-1)-1}(S)$ where α_1 is in the image of the J-homomorphism and $\langle \alpha_r, p_i, \alpha_s \rangle = \alpha_{r+s}$. He also found the elements $\beta_s \epsilon {}_p\pi_{2(sp+s-1)(p-1)-2}(S)$ with $\beta_1 = \langle \alpha_1, \alpha_1, \cdots (p) \cdots, \alpha_1 \rangle$. He proved that $\beta_1^p \alpha_1 \neq 0$ if and only if there is some nonzero element $\gamma \epsilon {}_p\pi_{2p^2(p-1)-2}(S)$. We prove that $\beta_1^p \alpha_1 \neq 0$, and combining this with some further advances of May [3] we have the following partial description of ${}_p\pi_*(S)$:

THEOREM. *The p-primary component of $\pi_n(S)$ for $0 < n \leqq (p^2+2p)q-6$ (where $q = 2(p-1)$) is described completely by the following table listing each element, its order, and its dimension*: (*In this table $j \geqq 1$.*)

Element	Order	Dimension
α_j	p	$jq-1$, $j \not\equiv 0 \pmod p$
$\alpha_{jp}^{(2)}$	p^2	$jpq-1$, $j \not\equiv 0 \pmod p$
$\alpha_{jp^2}^{(3)}$	p^3	jp^2q-1, $j \not\equiv 0 \pmod p$
$\beta_{m+1}\beta_1^{j-1}$	p	$((j+m)p+m)q-2j$, $0 \leqq m \leqq p-2$
$\alpha_1\beta_{m+1}\beta_1^{j-1}$	p	$((j+m)p+m+1)q-2j-1$, $0 \leqq m \leqq p-2$
γ	p	p^2q-2
$\beta_{m+1}\beta_1^{j-1}\gamma$	p	$(p^2+jp+mp+m)q-2j-2$, $0 \leqq m \leqq p-2$, $j=1$, if $m=1$ and $p=3$
$\alpha_1\beta_{m+1}\beta_1^{j-1}\gamma$	p	$(p^2+jp+mp+m+1)q-2j-3$, $0 \leqq m \leqq p-2$, $j=1$, if $m=1$ and $p=3$
$\alpha_m\gamma$	p	$(p^2+m)q-3$, $1 \leqq m \leqq p-2$
ϵ_m	p	$(p^2+m)q-2$, $1 \leqq m \leqq p-1$
ϕ	p^2	$(p^2+p)q-3$
$\beta_2\beta_{p-1}\beta_1^{j-1}$	p	$(p^2+jp+p-1)q-2j-2$; $j=1$ if $p=3$.

If we consider the above terms as being in $\pi_*(S) \otimes Z_p \subset \pi_*(L_p)$, then we have the following relations: (*Let $\alpha = \alpha_1$.*)

$$\alpha_n = (\alpha')^{n-1}\alpha, \quad \alpha_{np} = (\alpha')^{np-1}\alpha, \quad \alpha_{np^2} = (\alpha')^{np^2-1}\alpha, \quad n \not\equiv 0 \pmod p$$

$$\epsilon_m = (\alpha')^m\gamma + m(\alpha')^{m-1}\alpha\gamma', \quad 1 \leqq m \leqq p-1,$$

$$\beta_1'\beta_{p-1}' = \epsilon_{p-2}.$$

Furthermore, there are homotopy operations T and R such that $T(\beta_i) = \beta_{i+1}$, $1 \leqq i \leqq p-2$ and $R(\beta_1) = \gamma$. (Equalities hold for a particular choice of basis elements.)

REMARKS. By Toda [5], our result is equivalent to the following: Given N, consider the sequence of topological spaces $K_1 \subset K_2 \subset \cdots \subset S^N$ where $\pi_j(K_k) = 0$ for $j \geqq N+k$ and the inclusion $i: S^N \to K_k$ induces $i_*: \pi_j(S^N) \approx \pi_j(K_k)$ for $j < N+k$. Let $A^i(K_k; Z_p) = H^{N+i}(K_k; Z_p)$ for $0 \leqq i < N+k$. (This does not depend on N.) There is a generator $b_p \in A^{k+1}(K_k; Z_p)$ for $k = (p^2-1)q-2$. Then $\mathcal{P}^1 b_p \in A^{k+2p-1}(K_k; Z_p)$. Toda proves [5] that $\mathcal{P}^1 b_p = 0$ if and only if $\beta^p \alpha_1 \neq 0$. Thus it follows from our result that $\mathcal{P}^1 b_p = 0$.

BRIEF OUTLINE OF PROOF. We use the fact that d^r is known to be a derivation. The first nonzero element in positive dimension in the base is ξ_1 in $E^2_{q,0}$. Since it cannot be a boundary, it must be a noncycle. Hence there must be some nonzero element in $E^2_{0,q-1} \cong \pi_{q-1}(L_p)$. This element must be the image of α_1. Thus $d^q(\xi_1) = \alpha_1$. Then $d^q(\xi_1^2) = 2\xi_1\alpha_1, \cdots, d^q(\xi_1^{p-1}) = -\xi_1^{p-2}\alpha_1$, so that all these elements are cancelled (i.e. are boundaries or noncycles. But then $d^q(\xi_1^p) = p\xi_1^{p-1}\alpha_1 = 0$. Thus $\xi_1^{p-1}\alpha_1$ is not a boundary hence it must be a noncycle. Thus there must be a nonzero element in $E^2_{0,pq-2} \cong \pi_{pq-2}(L_p)$. (This element is β_1.) In exactly the same way, $\xi_1^{p-1}\beta_1^i\alpha_1$ is a nonboundary hence $\beta_1^{i+1} \neq 0$, $i \leqq p-1$. Then we assume that $\beta_1^p \alpha_1 = 0$. There is nothing natural to cancel β_1^{p+1} (which May [3] proves is nonzero). Thus it becomes increasingly difficult to assure that each noncycle is a boundary, and finally we see that there is an infinite cycle which is not a boundary. This contradicts the fact that $E^\infty_{s,t} = 0$ if $(s, t) \neq (0, 0)$, so the assumption is incorrect and $\beta_1^p \alpha_1 \neq 0$.

Detailed proofs and a more complete statement of our results will appear elsewhere.

REFERENCES

1. H. Cartan et al., *Algèbres d'Eilenberg-MacLane et homotopie*, Séminaire Henri Cartan de l'École Normale Supérieure, 17e année, (1954/55), 2eme ed., Secrétariat mathématique, 11 rue Pierre Curie, 1956.

2. S. Gitler and J. D. Stasheff, *The first exotic class of BF*, Topology **4** (1965), 257–266.

3. J. P. May, *The cohomology of the Steenrod algebra, stable homotopy groups of spheres*, Bull. Amer. Math. Soc. **71** (1965), 377–380.

4. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Princeton Univ. Press, Princeton, N. J., 1962.

5. H. Toda, *p-primary components of homotopy groups:* III, *Stable groups of the sphere*, Mem. Coll. Sci., Univ. Kyoto **31** (1958), 191–210.

6. ———, *Composition methods in homotopy groups of spheres*, Princeton Univ. Press, Princeton, N. J., 1962.

7. G. W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283.

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