REPRESENTATION THEORY OF CENTRAL TOPOLOGICAL GROUPS

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1. **Introduction.** We employ the notation and terminology introduced in the paper "On Central Topological Groups," p. 826 of this Bulletin. As announced there, the representation theory of $[Z]$-groups (as well as their structure theory) generalizes and unifies in a natural fashion that of compact groups on one hand and of locally compact abelian groups on the other.

The following basic definitions will be used throughout the exposition: (1) Let G be a topological group. Consider continuous finite-dimensional irreducible unitary representations ρ of G on the complex Hilbert space V_ρ; denote the degree of ρ by d_ρ and the identity map on V_ρ by I_{d_ρ}. Form equivalence classes of these representations, with respect to unitary equivalence, and choose one representation from each class. We denote by $\mathfrak{R}$ the totality of all such representations. (2) If $\rho\in\mathfrak{R}$ we denote by ρ_{ij} the coordinate functions associated with ρ relative to some orthonormal basis of V_ρ, by χ_ρ the character of ρ, and by $\mathfrak{X}$ the family of all such characters. (3) We denote by $\mathfrak{F}_c$, $\mathfrak{F}_u$, and $\mathfrak{F}_{c_0}$, respectively, the algebras of complex-valued functions on G which are continuous, uniformly continuous,[3] and continuous with compact support; by $\mathfrak{F}_r$ the subalgebra of $\mathfrak{F}_u$ consisting of the representative functions associated with representations in $\mathfrak{R}$, and by $\mathfrak{F}_z$ the subalgebra of $\mathfrak{F}_c$ consisting of the central functions. (4) If $f\in\mathfrak{F}_c$ and $x\in G$ then $x\triangle f$ denotes the conjugate of f by x, i.e., $(x\triangle f)(y) =f(xyx^{-1})$. The restriction of f to a subset S of G is f_S. If S is a subset on which f is bounded, $\|f\|_S$ stands for l.u.b. $\{|f(x)|/x\in S\}$. Finally, $\int_{G/Z}d\dot{x}$ denotes the normalized Haar integral on G/Z and $\int_G dx$ and $\int_Z dz$ are left invariant Haar integrals on G and Z respectively; normalized so that $\int_G=\int_{G/Z}\int_Z$; the associated Haar measures are denoted by $\mu_{G/Z}$, μ_G, and μ_Z. (5) At times, functions on G/Z will be regarded as functions on G.

The next two theorems are technical results required for the investigation.

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[3] Since $[Z]\subseteq[SIN]$, both uniform structures coincide.

THEOREM 1.1. *Assume that* $G\in[Z]$. *There exists a projection* $\#$ *from* $\mathfrak{F}_u$ *onto* $\mathfrak{F}_u\cap\mathfrak{F}_z$ *satisfying* $\|f^\#\|_F\leq\|f\|_F$ *for each compact invariant set* F *in* G. *Moreover, the subspaces* $\mathfrak{F}_r$ *and* $\mathfrak{F}_{c_0}$ *are invariant. This operator is defined by*

$$f^\#(y) = \int_{G/Z} (\dot{x} \triangle f)(y)\, d\dot{x}.$$

THEOREM 1.2. *If* $G\in[Z]$ *there is a nonnegative function* w *in* $\mathfrak{F}_{c_0}\cap\mathfrak{F}_z$ *with the property that, for any function* f *in* $L_1(G/Z)$,

$$\int_{G/Z} f(\dot{x})\, d\dot{x} = \int_G w(x)f(x)\, dx.$$

This is a weighting function.

We remark that it is possible to characterize $[SIN]$-groups in terms of function theory as groups having "small" central functions. This, in particular, gives that if $G\in[SIN]$ then $L_1(G)$ possesses an approximation to the identity consisting of central functions. Since $[Z]\subseteq[SIN]$ this means that $[Z]$-groups have this property.

2. **Finite-dimensionality of representations and an orthogonality relation.** A basic fact concerning representations of $[Z]$-groups is the following.

THEOREM 2.1. *Let* $G\in[Z]$ *and* ρ *be a weakly continuous irreducible unitary representation of* G *on the complex Hilbert space* V. *Then the following holds*:

(1) *For each choice of* u, v, u', v' *in* V, *the function defined by*

$$\dot{x} \to \langle \rho_x(u), v\rangle\langle\rho_x(u'), v'\rangle^-$$

is in $\mathfrak{F}_c(G/Z)$.

(2) ρ *is finite-dimensional and*

$$\int_{G/Z} \langle \rho_x(u), v\rangle\langle\rho_x(u'), v'\rangle^-\, d\dot{x} = d_\rho^{-1}\langle u, u'\rangle\langle v', v\rangle.$$

The proof of (2) rests upon a modification of a computation of Nachbin's [**14**] which yields the analogous fact for compact groups.

From the method of proof of Theorem 2.1 one recaptures a special case of a theorem on representations of discrete groups due to I. M. Isaacs and D. S. Passman [**9**].

3. **Separation and approximation theorems.** A direct consequence of Theorem 2.1 together with the theorem of Gelfand-Raikov (see

Naimark [15]) is that $\mathfrak{R}$ separates the points of G. Hence we have

THEOREM 3.1. *$[Z]$-groups are maximally almost periodic.*

In view of this theorem together with Corollary 1 of Theorem 2.3 of the preceding announcement, $[Z]$-groups satisfy the hypothesis of the Duality Theorem of S. Takahashi [18] which unifies the well-known duality theorems of Pontrjagin and Tannaka.

We remark that, with the help of the results of §5, Theorem 3.1 can be derived without appealing to the Gelfand-Raikov Theorem.

The following theorem summarizes the results concerning approximation of functions on $[Z]$-groups.

THEOREM 3.2. (1) *Each continuous function on G can be uniformly approximated on compact sets by representative functions.*

(2) *Each continuous central function can be uniformly approximated on compact sets by linear combinations of characters.*

(3) *$\mathfrak{X}$ separates the conjugacy classes of G.*

The proof of (1) follows from the above. In (2), crucial use is made of the # operator; (3) follows from (2).

4. The character formula and irreducibility criteria.

THEOREM 4.1. *Let $G\in[Z]$ and $\rho\in\mathfrak{R}$. Then χ_ρ is almost periodic and (for $f=\chi_\rho$) satisfies*

$$(*) \qquad f(s)f(t) = f(1)\int_{G/Z} f(xsx^{-1}t)\,d\dot{x} \qquad (s, t\in G).$$

Conversely, if f is any nontrivial continuous almost periodic function satisfying (*) then

$$\frac{f(s)}{f(1)} = \frac{\chi_\rho(s)}{\chi_\rho(1)},$$

for a unique ρ in $\mathfrak{R}$.

This generalizes the classical character formula for compact groups (see Weil [20]). For, as is easy to see, if G is compact and $f\in\mathfrak{F}_c(G/Z)$ then $\int_G f(x)dx=\int_{G/Z}f(\dot{x})d\dot{x}$, and elements of $\mathfrak{F}_c(G)$ are almost periodic. In addition, the theorem shows that in the case of an abelian group G a function f in $\mathfrak{F}_c(G)$ is a character of G if and only if it is almost periodic and $f/f(1)$ is a continuous homomorphism of G into the multiplicative group of **C**. The proof of Theorem 4.1 uses an extension of the # operator, as defined in §1, to representations, as well as the Fubini Theorem and several facts concerning the Bohr compactification.

THEOREM 4.2. *Let $G\in[Z]$ and ρ be a finite-dimensional continuous unitary representation of G. Then a necessary and sufficient condition for ρ to be irreducible is that either conditions* (1) *and* (2) *or conditions* (1) *and* (3) *hold, where* (1), (2), *and* (3) *are as follows.*

(1) $\int_{G/Z}\rho(xyx^{-1})d\dot{x}=(\chi_\rho(y)/\chi_\rho(1))I_{d_\rho}$ $(y\in G)$.

(2) $|\chi_\rho|^2$ *is a function on G/Z and* $\int_{G/Z}|\chi_\rho(x)|^2d\dot{x}=1$.

(3) $|\langle\rho_x(v),v\rangle|^2$ *is a function on G/Z and, for* $\|\dot{x}v\|=1$,

$$\int_{G/Z}|\langle\rho_x(v),v\rangle|^2\,d\dot{x}=d_\rho^{-1}.$$

COROLLARY. *Under the hypothesis of Theorem* 4.2, *ρ is a primary representation if and only if it satisfies* (1).

The following theorem generalizes a result of I. M. Isaacs and D. S. Passman [9] while relying on their method of proof as well as a number of our previous results.

THEOREM 4.3. *Let G be a $[Z]$-group which is nilpotent of class* 2. *If G has a faithful continuous finite-dimensional irreducible unitary representation $\rho: G\to U_{d_\rho}(\mathbf{C})$ then Z has finite index in G; in fact,* $[G:Z]=d_\rho^2$.

5. **Extension of group characters; representations of bounded degree.** In what follows, if H is an abelian topological group and $\chi: H\to \boldsymbol{T}$ a continuous homomorphism of H into the circle group $\boldsymbol{T}$ (written multiplicatively), we call χ a "group character" of H in order to distinguish from our previous use of the term "character."

DEFINITION. *Let G be a topological group, H an abelian subgroup, χ a group character of H, and ρ a finite-dimensional continuous representation of G. We say that ρ extends χ if $\rho(y)=\chi(y)I_{d_\rho}$, for all y in H.*

THEOREM 5.1. *Let G be a locally compact maximally almost periodic group and K a compact subgroup of G. Then each continuous unitary irreducible representation of K is* (*up to unitary equivalence*) *an irreducible component of the restriction ρ_K of a finite-dimensional continuous irreducible unitary representation ρ of G.*

COROLLARY 1. *Let G be a locally compact maximally almost periodic group and K be a compact central subgroup. Then each group character of K extends to a continuous finite-dimensional irreducible unitary representation of G.*

COROLLARY 2. *Let G be a locally compact maximally almost periodic group and K be a compact subgroup of G. Then the restriction map* $\mathfrak{F}_r(G) \to \mathfrak{F}_r(K)$ *is surjective.*

Both Theorem 5.1 and Corollary 2 are generalizations of facts known to hold for the case G compact.

In what follows we denote by G_1 the identity component of G. The following theorem generalizes a result of I. Kaplansky [**10**] on representations of bounded degree.

THEOREM 5.2. *Let G be an arbitrary locally compact group. If all the continuous irreducible unitary Hilbert space representations are finite-dimensional and of bounded degree then G_1 is abelian. Conversely, if G is a Lie group in $[Z]$ and G_1 is abelian then the continuous irreducible unitary representations of G are of bounded degree.*

The following theorem generalizes an important theorem of Pontrjagin [**16**] while relying on it.

THEOREM 5.4. *Let G be a $[Z]$-group and H a closed central subgroup of G. Then each group character χ of H extends to a continuous finite-dimensional irreducible unitary representation ρ of G. Moreover, if $x_0 \in G$, $x_0 \notin H$, then ρ can be chosen so that $\rho(x_0) \neq I_{d_\rho}$.*

Aside from Pontrjagin's theorem, the proof of this fact uses the structure theorem for $[Z]$-groups (Theorem 2.3 of the preceding announcement), Frobenius Reciprocity, and a theorem of Clifford [**2**].

THEOREM 5.5. *If G is a Lie group in $[Z]$ then there exists a continuous finite-dimensional unitary representation of G which is faithful on an open subgroup of $Z(G)$.*

In later work we will show that if, in addition, G is compactly generated, then it has a faithful continuous finite-dimensional unitary representation.

6. **An orthogonality relation and a criterion for equivalence of representations.** Since each continuous finite-dimensional irreducible unitary representation ρ of G has the property that $\rho(z) = \lambda(z)I$, where $z \in Z$ and λ is a group character of Z, this gives a map $\rho \to \lambda_\rho$, i.e., a map $\lambda: \mathfrak{R} \to \hat{Z}$, where $\hat{Z}$ denotes the character group of Z. Theorem 5.4 asserts that if $G \in [Z]$ then λ is surjective.

The next theorem gives an orthogonality relation analogous to one known to hold for compact groups; but it is less comprehensive than the latter.

THEOREM 6.1. *Let $G\in[Z]$ and $\rho, \sigma\in\mathfrak{R}$, where $\rho\neq\sigma$. If $\lambda_\rho=\lambda_\sigma$, where λ is the map defined above then the function $x\to\langle\rho_x(u), u'\rangle\langle\sigma_x(v), v'\rangle^-$, where $u, u'\in V_\rho$ and $v, v'\in V_\sigma$, is in $\mathfrak{F}_c(G/Z)$ and*

$$\int_{G/Z} \langle\rho_x(u), u'\rangle\langle\sigma_x(v), v'\rangle^- d\dot{x} = 0, \tag{1}$$

for all u, u', v, v'. In particular,

$$\int_{G/Z} \chi_\rho\bar{\chi}_\sigma\, d\dot{x} = 0. \tag{2}$$

We note that (1) can be stated

$$\int_G w(x)\langle\rho_x(u), u'\rangle\langle\sigma_x(v), v'\rangle^- dx = 0,$$

i.e., in view of Theorem 2.1,

$$\{d_\rho^{1/2}w^{1/2}\rho_{ij} \mid i, j=1, \cdots, d_\rho, \rho\in\lambda^{-1}(\chi)\},$$

for χ in $\hat{Z}$, is an orthonormal system in $L_2(G)$. In contradistinction to the case of compact G, more comprehensive orthogonality relations, in general, do not exist. In fact, the functions

$$\{d_\rho^{1/2}w^{1/2}\rho_{ij} \mid i, j=1, \cdots, d_\rho, \rho\in\mathfrak{R}\}$$

are orthonormal in $L_2(G)$ if and only if G is compact. It will appear later that this is related to a Plancherel formula for $[Z]$-groups, with explicit Plancherel measure.

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