# DISJOINT STEINER SYSTEMS ASSOCIATED WITH THE MATHIEU GROUPS

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A *Steiner system* of type *t-d-n* is a collection, $\mathfrak{D}$, of subsets of a set $S$ satisfying:

(i) The cardinality of $S$ is $n$.

(ii) Each subset in $\mathfrak{D}$ has cardinality $d$.

(iii) Every subset of $S$ of cardinality $t$ is contained in precisely one subset in $\mathfrak{D}$.

Here $t$, $d$, and $n$ are positive integers satisfying $t<d<n$.

Two Steiner systems, $S$, $\mathfrak{D}$ and $S'$, $\mathfrak{D}'$ are called *equivalent* if there is a bijection, $\phi: S \to S'$, such that $\phi(D) \in \mathfrak{D}'$ if and only if $D \in \mathfrak{D}$. If $S=S'$ and $\mathfrak{D}=\mathfrak{D}'$, the set of equivalences forms a group, called the *automorphism group* of the Steiner system $S$, $\mathfrak{D}$; it is a subgroup of the symmetric group on $S$. If $S$, $\mathfrak{D}$ and $S$, $\mathfrak{D}'$ are Steiner systems of the same type they are said to be *disjoint* if $\mathfrak{D} \cap \mathfrak{D}'$ is empty.

Among the most remarkable Steiner systems are the five associated with the Mathieu groups; i.e., those five Steiner systems whose automorphism groups are the five Mathieu groups. Witt [5] and [6] discussed them in detail, showing that they were unique up to equivalence.

Two of these five systems are of central importance. For if $S$, $\mathfrak{D}$ is a Steiner system of type *t-d-n* and $X$ a subset of $S$ of cardinality $h$ with $h<t$, then $S-X$, $\{D\cap(S-X) \mid X \subset D \in \mathfrak{D}\}$ is a Steiner system of type $(t-h)$-$(d-h)$-$(n-h)$. The two central Steiner systems referred to are of types 5-6-12 and 5-8-24. Their automorphism groups are, respectively, the Mathieu groups $M_{12}$ and $M_{24}$. The other three Steiner systems, of types 4-5-11, 4-7-23, and 3-6-22, are derived from these two by the above method; their automorphism groups are respectively, $M_{11}$, $M_{23}$, and $M_{22}$.

We have found a simple proof of the following

THEOREM. *There exist two disjoint Steiner systems of each of the above five types.*

It suffices to prove this for the two central systems of types 5-6-12 and 5-8-24. The proof, to be published elsewhere, uses a simple relationship between these Steiner systems and certain so-called error-correcting codes. That there are two systems of type 5-6-12 follows from the fact that the projective unimodular group of dimension 1 over GF(11) acts on a certain two distinct linear subspaces of the space of 12-tuples over GF(3). These subspaces are the error-correcting codes, each of which has associated with it a Steiner system of type 5-6-12. The group is transitive on each system. It follows that they are either disjoint or equal, and one proves immediately that they are not equal.

In the case of the Steiner system of type 5-8-24, elementary vector-space arguments reveal two disjoint systems once one knows the existence of the corresponding error-correcting codes. Again, by "excision" one gets two disjoint systems of type 4-7-23 and of type 3-6-22.

Observe that if $S$, $\mathfrak{D}$ and $S$, $\mathfrak{D}'$ are two disjoint Steiner systems, each of type $t$-$d$-$n$, then $S$, $\mathfrak{D}\cup\mathfrak{D}'=\mathcal{E}$ has the property that every subset of $S$ of cardinality $t$ is contained in precisely two subsets in $\mathcal{E}$. Such a system, $S$, $\mathcal{E}$, is called a *tactical configuration* of type 2; $t$-$d$-$n$. The question of the existence of tactical configurations of given types is, generally speaking, a very delicate one. As far as we know no one heretofore has demonstrated the existence of a tactical configuration of type 2; 5-8-24. It should be noted that the above method is the "hard" way to such a demonstration, since the tactical configuration is presumably more likely to exist than the two disjoint Steiner systems. The Steiner systems of type 5-6-12 and 5-8-24 together with the tactical configurations of type 2; 5-6-12 and 2; 5-8-24 are the only known tactical configurations with $t=5$.[1]

We do not know the answer to the obvious question: Is two the maximal number of disjoint Steiner systems of the above types? Cayley [1] showed, however, that there are two but no more disjoint Steiner systems of type 3-4-8; and his case bears rather close analogy to the two above that we have considered.

Part of our Theorem, namely, that there are two disjoint Steiner systems of type 5-6-12 (and hence of type 4-5-11), has been proved independently by D. R. Hughes [2] and [3].

Paige [4] discovered the connection between the Steiner systems of types 4-5-11 and 4-7-23 and certain so-called perfect error-correcting codes (though he did not use the latter term nor discuss disjointness of Steiner systems).

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[1] *Added in proof.* Implicitly known for some time has been the 48; 5-12-24 tactical configuration having $M_{24}$ as automorphism group. We have recently discovered several more tactical configurations with $t=5$ on 24 and 48 points.

## References

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2. D. R. Hughes, *On t-designs and groups*, Amer. J. Math. **87** (1965), 761–778.

3. ———, Private communication. This contains the disjointness proof, omitted from [**2**].

4. L. J. Paige, *A note on the Mathieu groups*, Canad. J. Math. **9** (1956), 15–18.

5. E. Witt, *Die 5-fach transitiven Gruppen von Mathieu*, Hamburg Abh. **12** (1936), 256–264.

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