ON SOME QUESTIONS IN NOETHERIAN RINGS

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1. **Introduction.** Goldie's theorem [1] establishes (among other things) that right Noetherian rings without nilpotent ideals have (right) classical quotient rings. In the general case the author [3] found a necessary and sufficient condition that an arbitrary right Noetherian ring have a right Artinian quotient ring. Using this criterion, it can be shown that a right hereditary, right Noetherian ring has a right Artinian quotient ring [2].

In this note we present an example of:

(1) a right and left Noetherian ring with a right, but not a left quotient ring and

(2) a right and left Noetherian ring with no quotient ring on either side.

2. **Notations and definitions.** From now on ring means ring with unit element, and Noetherian means right and left Noetherian. $N(R)$ denotes the maximal nilpotent ideal of a Noetherian ring R.

DEFINITION. If M is a subset of a ring A, then the *right annihilator* of M, $r(M)$, is $\{a \in A \mid Ma = 0\}$. We shall write $r_A(M)$ if there is a possibility of confusion about the ring. The *left annihilator* of M, $l(M)$, is defined analogously.

Recall that an element a in a ring R is *regular* if $r(a) = l(a) = 0$.

DEFINITION. A right (left) ideal I of a ring R is *essential* if I intersects every nonzero right (left) ideal of R nontrivially.

DEFINITION. $Z_r(R) = \{a \in R \mid r(a) \text{ is essential}\}$ is called the (right) *singular ideal* of R. $Z_l(R)$ is defined analogously.

3. **The examples.** Let $\boldsymbol{Z}$ denote the integers and $p \in \boldsymbol{Z}$ a prime. Define T to be the ring of all two-by-two "matrices" of the form:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \qquad a \in \boldsymbol{Z}, \qquad b \in \boldsymbol{Z}/(p), \qquad c \in \boldsymbol{Z}/(p),$$

where addition is component-wise and multiplication is given by:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}$$

where $\boldsymbol{Z}$ acts on $\boldsymbol{Z}/(p)$ in the usual way.

LEMMA 1. *T is Noetherian.*

PROOF. As an abelian group T is finitely-generated.

LEMMA 2. (i) *If*

$$r\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = 0,$$

then $a \notin (p)$. (ii) *Let*

$$x = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

then $l(x) = 0$ *and* $r(x) \neq 0$.

PROOF. For part (i), suppose $a \in (p)$, then

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.$$

Part (ii) is clear.

LEMMA 3. (i) $Z_l(T) \neq 0$.
(ii) $Z_r(T) = 0$.
(iii) *If* $t \in T$ *and* $r(t) = 0$, *then* $l(t) = 0$ *and* t *is regular.*

PROOF. Since T is Noetherian and $l(x) = 0$, Tx is an essential left ideal. But

$$Tx \subset l\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

It is well-known that the right singular ideal of a Noetherian ring is nilpotent. But,

$$N(T) = \left\{\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \middle| \, u \in \mathbf{Z}/(p)\right\}.$$

Thus, if $n \in Z_r(T)$, $r(n) \cap J = 0$, where

$$J = \left\{\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix} \middle| \, j \in \mathbf{Z}/(p)\right\}.$$

Finally, if $r(t) = 0$, then tT is essential, and if $t_1 t = 0$ we have $t_1 \in Z_r(T)$. So $l(t) = 0$.

We now note that the subring

$$T_1 = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a \in \mathbf{Z}, \; b \in \mathbf{Z}/(p)\right\}$$

is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}/(p)$. All the pieces are now available to construct the examples. The ring T has some amusing properties about which more will be said later.

Let S be the ring of all two-by-two "matrices" of the form

$$\begin{pmatrix} t & t' \\ 0 & t_1 \end{pmatrix},$$

$t, t' \in T$ and $t_1 \in T_1$. The ring operations are defined as above with T and T_1 acting on T by ordinary multiplication in T.

LEMMA 4. *S is Noetherian.*

PROOF. It is clear that S is finitely-generated as an abelian group. Let

$$y = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

where x is, as before, equal to

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in T_1.$$

LEMMA 5. *y is a regular element of S.*

PROOF. We show $l(y) = 0$; the other case is similar. Suppose

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} = 0.$$

Then, of course, $a = 0$, $bx = 0$ forces $b = 0$ as $l_T(x) = 0$ and $cx = 0$ yields $c = 0$ as x is regular in T_1.

LEMMA 6. *If*

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

is regular in S, then a is regular in T.

PROOF. By Lemma 3, it suffices to show that $r_T(a) = 0$. If not, there is a $t \neq 0$ such that $at = 0$. But, then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

We now can show that S does not satisfy the left Ore condition and, hence, S does not have a left quotient ring. Consider y and the element

$$\begin{bmatrix} 0 & \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & 0 \end{bmatrix}$$

where $c \notin (p)$.

THEOREM 1. *S does not satisfy the left Ore conaition.*

PROOF. If the Ore condition held, we would have an equation

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{bmatrix} 0 & \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & 0 \end{bmatrix},$$

where

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

is regular in S. By Lemma 6, this means that a is regular in T. The above equation forces

$$b_1 x = a \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}.$$

Writing this out fully we obtain

$$\begin{pmatrix} e_1 & f_1 \\ 0 & g_1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & f \\ 0 & g \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}.$$

Since a is regular, $e \notin (p)$. However, this last relation yields $e_1 p = ec \neq 0$. But, $e_1 p \in (p)$ and $ec \notin (p)$ as neither e nor c is, and a contradiction has been reached.

It can be shown that S has a *right* quotient ring. Let S^0 be the opposite ring of S. S^0 is again Noetherian, but S^0 does not possess a right quotient ring. Hence, it follows that

THEOREM 2. *$S \oplus S^0$ is Noetherian but satisfies neither Ore condition.*

Returning to S, we find that:

$$N(S) = \left\{ \begin{pmatrix} n & t \\ 0 & 0 \end{pmatrix} \middle| \, n \in N(T),\ t \in T \right\}.$$

Hence, $N(S)^3 = 0$ and $S/N(S) \approx T_1 \oplus T_1$. Therefore, given $s_1, s_2 \in S$ it follows that $(s_1 s_2 - s_2 s_1)^3 = 0$. S is, thus, a fairly well-behaved ring; indeed, it is a finitely-generated algebra over $\boldsymbol{Z}$. It also should be noted that the fact that S has a quotient ring on only one side while enjoying chain conditions on both is in marked contrast to the semi-prime case.

Finally, let us look a little at T. The following facts can be proved easily. T has global dimension two (T is Noetherian so there is no left-right problem), and T has a two-sided quotient ring which is not Artinian. This last fact shows that the result in [2] is, in some sense, best possible. Also, the existence of the element x with $l(x)=0$, but $r(x)\neq 0$ in a Noetherian ring with unit is of some interest.

References

1. A. W. Goldie, *Semi-prime rings with maximum condition*, Proc. London Math. Soc. (3) **10** (1960), 201–220.
2. L. W. Small, *Hereditary rings*," Proc. Nat. Acad. Sci. U.S.A. **55** (1966), 25–27.
3. ———, *Orders in Artinian rings*, J. Algebra, (to appear 1966).

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