HIGHER-DIMENSIONAL SLICE KNOTS

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The purpose of this paper is to demonstrate the existence of higher-dimensional smooth slice (or 0-concordant) knots of spheres in spheres with generalized Alexander polynomials which are not symmetric and which do not factorize. In particular, this provides a negative answer to questions B and C of Hirsch and Neuwirth [1, Part II]. The method is then extended to provide a generalization of the results of Levine [2].

1. **Algebraic theory.** Consider an Abelian group A which is finitely generated as a module over the group ring JZ of the infinite cyclic group $Z(t)$ (generated by t). An $m\times n$ matrix $M=(m_{ij}(t))$ whose entries are polynomials in t (integer coefficients) is said to present A as a module if there exists an exact sequence of JZ modules

$$F_2 \overset{d_2}{\longrightarrow} F_1 \longrightarrow A \longrightarrow 0$$

where F_1 and F_2 are free JZ modules on $(x_1, \cdots, x_n)$ and $(r_1, \cdots, r_m)$ respectively, and $d_2(r_i) = \sum_{j=1}^{n} m_{ij}(t)x_j$. See [3].

2. **Generalized Alexander polynomials.** A smooth n-knot is a smooth sphere pair (S^{n+2}, S^n). If $\pi_1(S^{n+2}-S^n)=G$, and $G'=$commutator subgroup of G, then the universal Abelian covering space $\tilde{X}$ of the knot complement $S^{n+2}-S^n$ is the regular covering space corresponding to G'. That is, $\pi_1(\tilde{X})=G'$, and the group of covering translations is the Abelian Group $G/G'=Z(t)$. The chain groups of $\tilde{X}$, and hence the homology groups $H_j(\tilde{X})$ are finitely generated as modules over $JG/G'=JZ(t)$, and have presentation matrices M_j, for all j (See [1]). In the terminology of [3], if $\epsilon_{1(j)}$ is the 1st elementary ideal of M_j and $\epsilon_{1(j)}$ is a principal ideal, then define the jth dimensional Alexander polynomial $\Delta_j(t)=$generator of $\epsilon_{1(j)}$. If M_j is square, then $\Delta_j(t) = |M_j|$, the determinant of M_j. When $j=1$, the polynomial is the usual Alexander polynomial [4, p. 353]. Also, one can easily define a whole sequence of generalized Alexander polynomials in dimension j, each one corresponding to a higher elementary ideal of the presentation matrix M_j (See [2].).

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3. The theorems. In the following, a smooth slice knot is a smooth sphere pair (S^{n+2}, S^n) which bounds a smooth ball pair (B^{n+3}, B^{n+1}).

THEOREM 1. *If* $n \geqq 2$, *and* $| \leqq j \leqq n/2$, *there exist smooth slice knots* (S^{n+2}, S^n) *with ith dimensional Alexander Polynomials* $\Delta_i(t)$, $1 \leqq i \leqq j$, *such that*:

(i) $\Delta_i(t) = 1 \quad 1 \leqq i < j$.

(ii) $\Delta_j(t)$ *is not symmetric* ($\Delta_j(t) \neq \pm t^\alpha \Delta_j(1/t)$, *any* α).

(iii) $\Delta_j(t)$ *does not factorize* (*there exists no polynomial* $F(t)$ *such that* $\Delta_j(t) = \pm t^\beta F(t) F(1/t)$, *some* β).

PROOF. The proof is based on the construction in [5] which gives knotted ball pairs in unknotted sphere pairs. The slice knots we are interested in will in all cases be the boundary of a constructed ball pair. In many cases the ball pairs will be precisely those obtained in [5], and the method of construction is exactly the same, but without the worry of embedding the ball pair in an unknotted sphere pair. The method may be described as follows: take an unknotted ball pair (B^{n+3}, B^{n+1}), and add a solid j-handle to $B^{n+3} - B^{n+1}$ by a trivial S^{j-1} embedded in $S^{n+2} - S^n = \partial(B^{n+3} - B^{n+1})$. Then if $K = (B^{n+3} - B^{n+1}) \cup h^j$, $K \simeq S^1 \vee S^j$ ($\simeq$ denotes homotopy equivalence) and $\partial K \simeq S^1 \vee S^j \vee S^{n+2-j}$. For $j > 1$ let $\alpha \in \pi_j(\partial K)$ represent the homotopy class of the inclusion map of S^j in $S^1 \vee S^j \vee S^{n+2-j}$, and t represent the action of the generator of $\pi_1(\partial K)$ on $\pi_j(\partial K)$. When $j = 1$, let α represent the path around the handle h^1 and β the path around the S^n. One adds a handle h^{j+1} whose attaching sphere S^j represents the element $2\alpha - t\alpha$ in $\pi_j(\partial K)(j > 1)$, or the element $\alpha^2\beta\alpha^{-1}\beta^{-1}$ in the case $j = 1$. This yields a knotted ball pair with boundary a knotted slice sphere pair.

The proof falls into two cases:

Case 1: $j = 1$. The construction gives a smooth slice knot (S^{n+2}, S^n) such that $\pi_1(S^{n+2} - S^n) = (\alpha, \beta | \alpha^2\beta\alpha^{-1}\beta^{-1})$. The Fox free derivative process [6] gives us $\Delta_1(t) = 2 - t$, when Abelianization sends α to 1 and β to t. This means that a result of Fox and Milnor [7] fails to generalize to higher dimensions, and gives us smooth slice knots which are not invertible and not $+$-amphicheiral (in the sense of [6]). Also, it proves that Neuwirth's group of knot groups [8, Chapter 8], is nontrivial for $n \geqq 2$.

Case 2: $j > 1$. The construction can again be used to produce a smooth slice knot (S^{n+2}, S^n) such that $\pi_1(S^{n+2} - S^n) = Z(t)$, $\pi_i(S^{n+2} - S^n) = 0$, $1 < i < j$, and $\pi_j(S^{n+2} - S^n) = (t^i\alpha | 2\alpha - t\alpha)$. This means that $\pi_1(\tilde{X}) = G' = 0$, and the Hurewicz Isomorphism Theorem implies $H_j(\tilde{X}) = \pi_j(\tilde{X}) = \pi_j(S^{n+2} - S^n)$. The matrix presenting $H_j(\tilde{X})$ as a module over $JG/G' = JZ(t)$ is the 1×1 matrix $(2-t)$, so $\Delta_j(t) = 2-t$. When $j = n/2$ (n even) this answers questions B and C of Hirsch and Neuwirth [1] in the negative.

In fact, the construction allows us to prove the following:

THEOREM 2. *For $n \geqq 2$, $1 \leqq j \leqq n/2$, and given any polynomial $F(t)$ such that $F(1) = \pm 1$, then there exists a smooth slice knot (S^{n+2}, S^n) with Alexander polynomials $\Delta_i(t)$, $1 \leqq i \leqq j$, such that $\Delta_j(t) = F(t)$.*

PROOF. The construction and reasoning is exactly the same as in Theorem 1, with the polynomial $F(t)$ becoming a relation in the appropriate homotopy group.

Case 1: $j > 1$. Construct the knot as in Theorem 1, but attach h^{j+1} by the element $F(t)\alpha$ instead of $2\alpha - t\alpha$. This yields a slice knot such that $\pi_1(S^{n+2} - S^n) = Z(t)$, $\pi_i(S^{n+2} - S^n) = 0$, $1 < i < j$, and $\pi_j(S^{n+2} - S^n) = (t^i\alpha \mid F(t)\alpha)$. Clearly $\Delta_j(t) = F(t)$.

Case 2: $j = 1$. (I would like to thank Dr. J. F. P. Hudson for pointing out the validity of the theorem in this case.) Given a polynomial $F(t) = \sum_{i=0}^{m} a_i t^i$, construct the knot as in Theorem 1, adding the second handle by the element $\alpha^{a_0}\beta\alpha^{a_1} \cdots \beta\alpha^{a_m}\beta^{-m}$ instead of $\alpha^2\beta\alpha^{-1}\beta^{-1}$. This yields a slice knot such that

$$\pi_1(S^{n+2} - S^n) = (\alpha, \beta \mid \alpha^{a_0}\beta\alpha^{a_1} \cdots \beta\alpha^{a_m}\beta^{-m}).$$

The Fox free derivative process yields $\Delta_1(t) = F(t)$, where $\alpha \to 1$ and $\beta \to t$ as before. When $n = 2$, this provides a very simple proof of the results of Kinoshita [9], who constructed a smooth (not necessarily slice) sphere pair (S^4, S^2) corresponding to the given polynomial $F(t)$. One could also compare the results of Terasaka (see [6, p. 136]), who constructed a smooth slice knot (S^4, S^2) corresponding to any factorizing polynomial.

Suppose now that we are given a sequence of polynomials $F_i(t)$, $1 \leqq i$, such that

(a) for some integer p, $F_i(t)$ is a unit (i.e. $F_i(t) = \pm t^\alpha$) for $i > p$.

(b) $F_i(1) = \pm 1$.

(c) $F_{i+1} \mid F_i$.

(d) If $\lambda_i = F_{i-1}/F_i$, then $\lambda_{i+1} \mid \lambda_i$.

Then we have the following generalization of the results of Levine [2]:

THEOREM 3. *For $n \geqq 2$, $1 \leqq j \leqq n/2$, and given any sequence of polynomials F_i satisfying* (a)–(d), *then there exists a smooth slice knot (S^{n+2}, S^n) with the F_i as its sequence of generalized Alexander polynomials in dimension j.*

PROOF. Reasoning as in [2], the problem reduces to finding a slice knot which has a diagonal matrix with entries λ_i presenting $H_j(\tilde{X})$. Theorem 2 tells us that we can find a slice knot for each nonunit λ_i since $\lambda_i(1) = \pm 1$, and we take the connected sum of all these knots to produce the desired one.

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