RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

NORMAL OPERATORS, LINEAR LIFTINGS AND THE WIENER COMPACTIFICATION[1]

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Recently A. S. Galbraith communicated to the authors the conjecture that normal operators (Nakai-Sario [4]) are linear liftings (Tulcea [7]). In the present Research Announcement we shall show that the conjecture is correct: conditions (1)–(5) of [4] imply conditions (I)–(V) of [7].

Galbraith's conjecture also led us to a generalization of normal operators where we make use of Wiener's compactification of a Riemann surface (cf. Constantinescu-Cornea [2]). The results have applications to Ahlfors' [1] conjecture on extreme normal operators.

We wish to express our sincere gratitude to Dr. Galbraith for stimulating this research.

1. **Normal operators on Wiener's boundary.** Consider a finite union $R=\bigcup_{j=1}^n R_j$ of disjoint hyperbolic Riemann surfaces R_j with Wiener harmonic boundaries Γ_j (Constantinescu-Cornea [2]). Decompose Γ_j into two disjoint compact sets α_j and β_j, the case $\beta_j=\emptyset$ (void) not excluded. Set $\Gamma=\bigcup_1^n \Gamma_j$, $\alpha=\bigcup_1^n \alpha_j$ and $\beta=\bigcup_1^n \beta_j$.

We are interested in mappings of the space $C(\alpha)$ of real-valued continuous functions on α into the space $H(R)$ of harmonic functions on R. An operator L from $C(\alpha)$ into $H(R)$ is, by definition, *normal if* (L.1) *L is linear*, (L.2) $f\geqq 0$ *implies* $Lf\geqq 0$, (L.3) $Lf|\alpha=f$, (L.4) $L1=1$, *and* (L.5) $\int_\gamma {}^*dLf=0$ *along a dividing cycle γ on R homologous to α.*

Clearly $Lf\in HB(R)$, the space of bounded functions in $H(R)$. If R is a bordered Riemann surface with compact border $\bar{\alpha}$, and if α lies on $\bar{\alpha}$, then L is normal in the original sense [4], as stated more precisely in §4 below.

In the case $\beta=\emptyset$, an operator L satisfying (L.1) and (L.2) gives

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$Lf = H_R^f$, the unique harmonic function on R with boundary values f on $\alpha = \Gamma$. Conditions (L.3)–(L.5) are trivially satisfied, and there exists one and only one normal operator. For this reason our main interest is with the case $\beta \neq \emptyset$.

2. **Operators between function spaces.** For a normal operator L, $f \to Lf|\beta$ gives an operator T_L from $C(\alpha)$ into $C(\beta)$. Conversely, given an operator T from $C(\alpha)$ into $C(\beta)$, there exists a unique operator L_T from $C(\alpha)$ into $HB(R)$ such that $f \to L_T f|\beta$ is T. We shall call T *normal* if L_T is normal.

We wish to characterize normal operators among operators from $C(\alpha)$ into $C(\beta)$. Let $K(z, p)$ be the Wiener harmonic kernel on $R \times \Gamma$ and let μ be the harmonic measure on Γ (see [3]). We consider the measure ν on Γ given by

$$d\nu(p) = \left(\int_\gamma {}^*d_z K(z, p) \right) d\mu(p),$$

where γ is as in (L.5). Clearly ν depends only on μ and on the decomposition of Γ into α and β.

An operator T from $C(\alpha)$ into $C(\beta)$ is normal if and only if (T.1) *T is linear,* (T.2) *$f \geqq 0$ implies $Tf \geqq 0$,* (T.3) *$T1 = 1$, and* (T.4) $\int_\alpha f d\nu = \int_\beta Tf d\nu$.

3. **Operators between measure spaces.** The totality $\{T\}$ of normal operators T from $C(\alpha)$ into $C(\beta)$ forms a convex set. A generalized form of Ahlfors' problem [1] is to determine the set $E\{T\}$ of extreme points of $\{T\}$. In this connection it is also interesting to investigate the conjugate operator T^* of $T \in \{T\}$. Let $M(\alpha)$ (resp. $M(\beta)$) be the conjugate space of $C(\alpha)$ (resp. $C(\beta)$), i.e., the totality of signed regular Borel measures σ on α (resp. β). Then T^* is an operator from $M(\beta)$ into $M(\alpha)$ given by $\int_\beta Tf d\sigma = \int_\alpha f dT^*\sigma$ for $f \in C(\alpha)$ and $\sigma \in M(\beta)$. Among operators from $M(\beta)$ into $M(\alpha)$, we shall again call T^* *normal* if it corresponds to a normal T.

An operator T^* of $M(\beta)$ into $M(\alpha)$ is normal if and only if (T*.1) *T^* is linear,* (T*.2) *$\sigma \geqq 0$ implies $T^*\sigma \geqq 0$,* (T*.3) $\int_\alpha dT^*\sigma = \int_\beta d\sigma$, (T*.4) *$T^*\nu = \nu$, and* (T*.5) *$T^*$ is weakly continuous.*

The determination of $E\{T^*\}$ is equivalent to that of $E\{T\}$. In the case where α and β are homeomorphic and the homeomorphism preserves ν, Ahlfors' conjecture [1] can be restated as follows: $T^* \in E\{T^*\}$ if and only if there exists a homeomorphism $i = i_{T^*}$ of β onto α such that $T^*\epsilon_p = \epsilon_{i(p)}$, where ϵ_p is a point mass at $p \in \beta$. This is certainly the case for operators with (T*.1)–(T*.4). However, condition (T*.5) shows that the conjecture is not universally valid. Actually, Savage [5] gave a counter example even in the simple case of

R: $1<|z|<2$, with α and β lying over $|z|=1$ and 2, respectively.

There exist, of course, cases where Ahlfors' conjecture is correct. Take, e.g., a Riemann surface $F \in O_{HB}^n - O_{HB}^{n-1}$ with $n=1, 2, \cdots$ (see Constantinescu-Cornea [2]), remove a disk F_0, and form the double R of $F-\bar{F}_0$ about ∂F_0. As α we take the Wiener harmonic boundary of F, and as β, the symmetric image of α in R. Then $\alpha=\beta$ is a set of n isolated points. If we take the center of μ on ∂F_0, then $\nu|\alpha$ and $\nu|\beta$ are symmetric and atomic, and (T*.5) is immediate.

4. **Normal operators on the relative boundary.** As a special concrete case we consider the complement R of a regular region of an open Riemann surface, with α lying on the relative boundary $\bar{\alpha}$ of R. There exists a unique continuous mapping π of $R\cup\alpha\cup\beta$ onto $R\cup\bar{\alpha}\cup\beta$ such that π is an identity on $R\cup\beta$, and $\bar{\omega}=(\mu|\alpha)\circ\pi^{-1}$, where $\bar{\omega}$ is the harmonic measure on $\bar{\alpha}$ with respect to $R\cup\bar{\alpha}\cup\beta$.

A normal operator $\bar{L}$ of $C(\bar{\alpha})$ into $H(R)$ in the original sense [4] can be represented as an operator $\bar{T}$ of $C(\bar{\alpha})$ into $C(\beta)$ with (T.1)–(T.4). Here $R\cup\alpha\cup\beta$ and ν are replaced by $R\cup\bar{\alpha}\cup\beta$ and a suitable $\bar{\nu}$ such that $\bar{\nu}=(\nu|\alpha)\circ\pi^{-1}$ on $\bar{\alpha}$ and $\bar{\nu}=\nu|\beta$ on β. We may take $C(\alpha)=L^\infty(\bar{\alpha}, \bar{\omega})$, which contains $C(\bar{\alpha})$ as a subspace.

For any normal operator $\bar{T}$ of $C(\bar{\alpha})$ into $C(\beta)$, there exists a $T\in\{T\}$ such that $T|C(\bar{\alpha})=\bar{T}$, and vice versa.

5. **Linear liftings.** Let $Z=\bar{\alpha}\cup\beta$ and ω be a measure on Z such that $\omega|\bar{\alpha}=\bar{\omega}$ and $\bar{\omega}|\beta=0$. For $\bar{L}$ and $f\in L^\infty(Z, \omega)$ we set $\rho f=f|\bar{\alpha}$ on $\bar{\alpha}$ and $\rho f=\bar{L}(f|\alpha)|\beta$. Then ρ is a linear lifting on $L^\infty(Z, \omega)$ (Tulcea [7]). Thus a normal operator gives an example of a linear lifting. Ahlfors' problem [1] can then be stated as follows: when is a linear lifting a lifting, i.e., a multiplicative linear lifting ?

References

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