STABLE COMPLEX MANIFOLDS[1]

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1. T. Van de Ven [3] has recently shown that there exist real 4-dimensional manifolds which admit almost complex structures but admit no complex structures, e.g. $S^1 \times S^3 \# S^1 \times S^3 \# CP(2)$. The purpose of this note is to show that this is an unstable phenomenon.

Let M^n be a C^∞ n-dimensional real manifold without boundary and let τ_M be its tangent bundle. R^k is real Euclidean k-space and C^k is complex k-space.

DEFINITION 1. M^n admits a *stable complex structure* if $M^n \times R^k$ can be given the structure of a complex analytic manifold for some $k \geqq 0$, $n \equiv k \pmod 2$.

Let ξ^m be an m-plane bundle over M^n.

DEFINITION 2. A *stable complex structure for* ξ^m is a reduction of the group of $\xi^m \oplus \epsilon^k$ to $U((m+k)/2)$ for some $k \geqq 0$, $m \equiv k \pmod 2$.

DEFINITION 3. A *stable almost complex structure for* M^n is a stable complex structure for τ_M.

PROPOSITION. *M^n admits a stable complex structure if and only if it admits a stable almost complex structure.*

2. It is clear that a stable complex structure carries with it a stable almost complex structure. We show the converse is true.

We can assume M^n is a real analytic manifold. There exists a complex n-dimensional manifold N_c^n (of real dimension $2n$) and a real analytic embedding $i: M^n \subset N_C^n$ [4]. Regarding N as a real manifold, it is easy to see from the construction of [4] that $\nu(i)$, the normal bundle of the embedding, is equivalent to τ_M; i.e., $\tau_N | M \approx \tau_M \oplus \tau_M$. Let $U \subset N$ be a tubular neighborhood of M in N and let $r: U \to M$ be the bundle projection (U is identified with the total space of $\nu(i)$). One can construct an open neighborhood, V, of M in U $(M \subset V \subset U \subset N)$ which is a domain of holomorphy [2]. Let $r: V \to M$ be the restriction of $r: U \to M$.

Let ν^{2n+1} be the stable normal bundle to M^n. $\tau_M \oplus \nu^{2n+1} \approx \epsilon^{3n+1}$. τ_M admits a stable complex structure if and only if ν does. Assume now that there is a bundle σ over M with fiber C^m and group $U(m)$

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and an R^{2m}-bundle equivalence $f: \sigma \to \nu \oplus \epsilon^k$, $2n+k+1=2m$. We have $r^*\sigma$ over V. By Satz II of [1], we can assume $r^*\sigma$ is a holomorphic vector bundle over V. Let $\Pi: E \to V$ be the bundle projection. Considering E as a real analytic manifold we have

$$\begin{aligned} \tau_E \mid M &\approx \Pi^*\tau_V \mid M \oplus \Pi^* r^*\sigma \mid M \\ &\approx \tau_V \mid M \oplus r^*\sigma \mid M \\ &\approx \tau_N \mid M \oplus \sigma \\ &\approx \tau_M \oplus \tau_M \oplus \nu \oplus \epsilon^k \\ &\approx \tau_M \oplus \epsilon^{3n+k+1}. \end{aligned}$$

Therefore the embedding $M \subset V \subset E$ has a trivial normal bundle. $M^n \times R^{3n+k+1}$ can now be regarded as an open subset of E and so it inherits the structure of a complex manifold from the complex structure on E. Q.E.D.

References

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