

SUMS OF ULTRAFILTERS

BY ZDENĚK FROLÍK

Communicated by E. Hewitt, June 6, 1966

The main result, an estimate of the cardinality of the set of all ultrafilters producing a given type of ultrafilter (see definition 1.4 and Theorem C in 1.4), is illustrated by a proof of nonhomogeneity of $\beta N - N$ (see 2.1) without using the continuum hypothesis, and by an exhibition of the following two examples.

THEOREM A. *For each positive integer n there exists a space X such that X^n is countably compact but X^{n+1} is not.*

THEOREM B. *There exists a space Y such that each finite product Y^n is countably compact but Y^{\aleph_0} is not.*

By a space we mean a separated uniformizable topological space; and Z^m stands for the product of any constant family $\{Z | a \in A\}$ such that the cardinal of A is m .

In our examples the spaces X^{n+1} in A and Y^{\aleph_0} in B are not pseudocompact. An exhibition of A and B with countably compact replaced by pseudocompact is done in [3]; it does not require Theorem C. Trivial examples of spaces with properties in A and B do not seem to be available.

Observe the proof of A and B reduces to the following.

THEOREM A'. *For each positive integer n there exist spaces $X(1), \dots, X(n+1)$ such that the product of any family $\{X(k_i) | i=1, \dots, n\}$ is countably compact but the product $\{X(j) | j \leq n+1\}$ is not countably compact.*

THEOREM B'. *There exists a sequence $\{Y(j)\}$ of spaces such that the product of any finite subfamily is countably compact but the product of $\{Y(j)\}$ is not.*

Indeed, for an X in A take the sum of spaces $X(j)$ with properties in A'. For Y in B take a one-point countable-compactification of the sum of a sequence $\{Y(j)\}$ with properties in B'; then the product of $\{(j) \times Y(j)\}$ is a closed subspace of Y .

REMARK. In addition, we shall exhibit $\{Y(j)\}$ such that the product of any proper subfamily (e.g. $\{Y(j) | j \geq 2\}$) is countably compact. On the other hand there exists a sequence $\{Y(j)\}$ such that the product of a subfamily is countably compact if and only if the subfamily is finite.

In what follows N denotes the set and the discrete space of natural numbers, βN the Stone-Čech compactification of N such that $N \subset \beta N$ and the points of $\beta N - N$ are free ultrafilters on N , $N^* = \beta N - N$, P the set of all permutations of N , f^* with f in P the continuous extension of f to a mapping (homeomorphism) of βN onto itself, and P^* the set of all f^* with f in P .

1. Types and production of types. Recall that the cardinal of βN is $\exp \exp \aleph_0$, the cardinal of every dense set in N^* is at least $\exp \aleph_0$, and any discrete countable subset of βN is normally embedded in βN .

1.1. Let T be a set and τ be a mapping of $N^* = \beta N - N$ onto T such that $\tau x = \tau y$ if and only if $p^*x = y$ for some p in P . The elements of T are called the types of ultrafilters on N , and if $t = \tau x$ then t is called the type of x and x is said to be of type t . The set $\tau^{-1}[(t)]$ of all $x \in N^*$ of type t is of cardinal $\exp \aleph_0$ because the cardinal of P is $\exp \aleph_0$ and $\tau^{-1}[(t)]$ is clearly dense in N^* . It follows that the cardinal of T is $\exp \exp \aleph_0$.

If M is any countable infinite set and $f: M \rightarrow N$ is a bijective mapping then the type of any free ultrafilter x on M is defined to be the type of the "image" of x under f . Clearly the definition does not depend on f .

1.2. If X is any collection of ultrafilters on a set M and if y is an ultrafilter on X then the sum of X , with respect to y designated by $\sum_y X$, is defined to be the collection z of all sets of the form $\bigcup \{M_x | x \in Y\}$ where $Y \in y$ and $M_x \in x$ for each x in Y . It is easy to show that z is actually an ultrafilter on M . A collection X of ultrafilters is called to be discrete if there exists a disjoint family $\{M_x | x \in X\}$ with $M_x \in x$. In a natural way we apply those definitions to a one-to-one family $\{x_\alpha\}$ of ultrafilters and an ultrafilter on the index set.

Now let $\{x_n\}$ and $\{x'_n\}$ be two discrete sequences of ultrafilters on a countable set M such that $\tau x_n = \tau x'_n$ for each n . Then the sums $\sum_y \{x_n\}$ and $\sum_y \{x'_n\}$ are of the same type for any ultrafilter y on N , and so we may introduce the following definition.

1.3. DEFINITION. If $\{t_n\}$ is any sequence of types and y is any ultrafilter on N then the sum t of $\{t_n\}$ with respect to y is designated by $\sum_y \{t_n\}$ and defined to be the type of any $\sum_y \{x_n\}$ with $\{x_n\}$ a discrete sequence of ultrafilters such that $\tau x_n = t_n$. It is clear that then any x of type t is of the form $\sum_y \{x_n\}$.

Now we are prepared to introduce the main concept—the producing relation.

1.4. DEFINITION. The producing relation ϕ on T is defined to be the set of all pairs $\langle u, v \rangle$ such that $v = \sum_y \{t_n\}$ for some y of type u

and some sequence $\{t_n\}$ in T . Thus, the domain of ϕ is T , its range is contained in T and $\phi[(t)] = E\{\sum_v \{t_n\} | y \in t\}$. The symbol $\langle u, v \rangle \in \phi$ will often be read either “ u produces v ” or “ v is produced by u .” The main result reads as follows.

THEOREM C. *Any type is produced by at most $\exp \aleph_0$ types, and any type produces $\exp \exp \aleph_0$ types, i.e. $\text{card } \phi^{-1}[(t)] \leq \exp \aleph_0, \text{card } \phi[(t)] = \exp \exp \aleph_0$ for any type t .*

The proof will be given in 1.7 and 1.9 below after we develop a topological interpretation of the relation ϕ .

1.5. It is easy to see that a countable $X \subset N^*$ is a discrete subset of the topological space βN if and only if X is discrete in the sense of 1.2, and that $\text{cl } X$ is homeomorphic to βN if X is infinite. Thus given a z in $\text{cl } X - X$, the traces of neighborhoods of z on X form an ultrafilter z_X on X whose type will be denoted by τ_{Xz} and called the type of z relative to X . Clearly $z = \sum_{z_X} X$ and so $\langle \tau_{Xz}, tz \rangle \in \phi$. If y is any free ultrafilter on X then $z = \sum_v X$ belongs to $\text{cl } X - X$ and $y = z_X$.

Now let $\{t_n\}$ be any sequence of types and t be a type. Choose a discrete sequence $\{x_n\}$ of representatives (that means $\tau x_n = t_n$), and consider the set of all x_n . The set of all $\sum_v \{t_n\}, \tau y = t$, coincides with the set $E\{\tau z | z \in \text{cl } X - X, \tau_{Xz} = t\}$. So $\phi[(t)] = E\{\tau z | z \in \text{cl } X - X, \tau_{Xz} = t \text{ for some discrete countable } X \subset N^*\}$. In what follows we shall use that topological interpretation without any reference.

1.6. Let $\{M_n\}$ be a countable decomposition of N and let $x_n, y_n \in \text{cl } M_n - M_n$. If $x_n \neq y_n$ for each n then $\text{cl } E\{x_n\} \cap \text{cl } E\{y_n\} = \emptyset$ and conversely. The proof is evident.

1.7. The second statement of Theorem C follows immediately from 6. Indeed taking any decomposition $\{M_n\}$ of N with all M_n infinite, we have $\text{card } \text{cl } M_n - M_n = \exp \exp \aleph_0$ for each n and therefore we get $\exp \exp \aleph_0$ disjoint sets $\text{cl } X = \text{cl } E\{x_n\}$ each containing at least one point y with $\tau_X y = t$, which is of type in $\phi[(t)]$. Since $\text{card } \tau^{-1}[(t)] = \exp \aleph_0$, the result follows.

To prove the first statement of Theorem C we need the following lemma.

1.8. Let $y \in \beta N - N$. There exists a set \mathfrak{X} , $\text{card } \mathfrak{X} \leq \exp \aleph_0$, of discrete countable subsets X of $\beta N - N$ such that if Y is any discrete countable subset of $\beta N - N$, and if $y \in \text{cl } Y - Y$, then $Y \supset X$ for some $X \in \mathfrak{X}$.

PROOF. For each countable decomposition $\{M_n\}$ of N choose an $x_n \in \text{cl } M_n - M_n$ such that $y \in \text{cl } X - X$, if possible, and take all $X' \subset X$ with $y \in \text{cl } X' - X$. The set \mathfrak{X} of all X' , $\{M_n\}$ variable, has

required properties by 1.6. The cardinal of \mathfrak{X} is at most $\exp \aleph_0 \exp \aleph_0 = \exp \aleph_0$.

1.9. To prove the first statement of Theorem C it now suffices to combine 1.7 with 1.5 and the following simple observation: If $X \subset Y$, $y \in \text{cl } X - Y$ and Y is a discrete subset of $\beta N - N$, then $\tau_X y = \tau_Y y$, i.e. τ_X is a restriction of τ_Y .

THEOREM C'. *Let $\phi_\infty = \bigcup \{ \phi^k \mid k \in \mathbb{N}, k \neq 0 \}$, $\phi_\infty^{-1} = \bigcup \{ (\phi^{-1})^k \mid k \in \mathbb{N}, k \neq 0 \}$, where $\rho^k = \rho \circ \dots \circ \rho$ (k -times). Then $\phi_\infty^{-1} = (\phi_\infty)^{-1}$, and $\text{card } \phi_\infty[(t)] = \exp \exp \aleph_0$, $\text{card } \phi_\infty^{-1}[(t)] \leq \exp \aleph_0$ for any t in T .*

2. Applications. A space is called homogeneous if any point can be mapped onto any point by an autohomeomorphism. W. Rudin proved in [5] that the space N^* is not homogeneous by proving the existence of the so called P -points. His proof of the existence of P -points heavily depends on the continuum hypothesis. Theorem C enables us to prove the nonhomogeneity of N^* without the continuum hypothesis.

2.1. Proof of nonhomogeneity of N^* . For each x in N^* denoted by T_x the set of all relative types of x ; i.e. $T_x = \phi^{-1}[\tau x]$. If $lx = y$ for some autohomeomorphism l of N^* , then clearly $T_x = T_y$. Since the sets T_x are of the cardinals at most $\exp \aleph_0$, $\text{card } T = \exp \exp \aleph_0$ and $\{T_x \mid x \in N^*\}$ is a covering of T , the result follows.

REMARK. It should be remarked that we have proved the existence of $\exp \exp \aleph_0$ equivalence classes. Those equivalence classes define "free types" of free ultrafilters. Relative free types are defined similarly.

It remains to prove Theorems A' and B'.

2.2. **LEMMA.** *There exists a disjoint transfinite family $\{T_\alpha \mid \alpha < \omega_1\}$ of subsets of T and a family $\{t_\alpha \mid \alpha < \omega_1\}$, $t_\alpha \in T$, such that, denoting by X_α the set of all points of βN of types in T_α , each countable discrete subset X of $\bigcup \{X_\beta \mid \beta < \alpha\}$ has a cluster point in X_α of type t_α with respect to X .*

THEOREM D. *For any set A of countable ordinals let $P_A = N \cup \{X_\alpha \mid \alpha \in A\}$ be a subspace of βN . If $\{A_b \mid b \in B\}$ is a countable family of sets of countable ordinals, then the product $P = \prod \{P_{A_b} \mid b \in B\}$ is countably compact if $\bigcap \{A_b\}$ is unbounded, and it is not countably compact if that intersection is empty.*

First we prove Theorem D, then Theorems A', B', and finally the main step, the Lemma.

2.3. **PROOF OF THEOREM D.** If the intersection is empty then the "diagonal" is a closed infinite discrete subspace of the product. For

