ORLICZ SPACES OF FINITELY ADDITIVE SET FUNCTIONS, LINEAR OPERATORS, AND MARTINGALES[1]

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The purpose of this note is to announce some properties and applications of Orlicz spaces of finitely additive set functions, the V^Φ spaces. The V^Φ spaces are natural generalizations of the V^p spaces (Bochner [2] and Leader [6]).

1. **The $V^\Phi(\mathfrak{X})$ spaces.** Throughout this note Ω is a point set, Σ a field of subsets of Ω, μ a finitely additive extended real valued nonnegative set function defined on Σ; and $\Sigma_0 \subset \Sigma$ is the ring of sets of finite μ-measure. A partition π is a finite disjoint collection $\{E_n\} \subset \Sigma_0$. The partitions are partially ordered by defining $\pi_1 \leqq \pi_2$ whenever each $E_n \in \pi_1$ is a union of members of π_2. $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach (or B-) spaces with conjugate spaces $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ respectively. Φ is a (nontrivial) Young's function with complementary function Ψ.

DEFINITION. $V^\Phi(\Omega, \Sigma, \mu, \mathfrak{X}) = (V^\Phi(\mathfrak{X}))$ consists of all finitely additive μ-continuous $\mathfrak{X}$-valued set functions F on Σ_0 such that for some $k > 0$,

$$I_\Phi(F/k) = \sup_\pi \sum_\pi \Phi\left(\frac{\|F(E_n)\|}{k\mu(E_n)}\right)\mu(E_n) \leqq 1,$$

where the supremum is taken over all partitions $\pi = \{E_n\}$ and the convention $0/0 = 0$ is observed.

$V^\Phi(\mathfrak{X})$ becomes a B-space under each of the equivalent norms

$$N_\Phi(F) = \inf\{k > 0: I_\Phi(F/k) \leqq 1\}$$

or

$$\|F\|_\Phi = \sup\left\{\sup_\pi \sum_\pi \frac{\|F(E_v)\|\,\|G(E_v)\|}{\mu(E_v)} : G \in V^\Psi(\mathfrak{X}^*),\ N_\Psi(G) \leqq 1\right\}.$$

Using the integration procedure of [4, Chap. III], one can define the (possibly incomplete) Orlicz spaces $L^\Phi(\Omega, \Sigma, \mu, \mathfrak{X}) (= L^\Phi(\mathfrak{X}))$ of totally μ-measurable $\mathfrak{X}$ valued functions f satisfying $\int_\Omega \Phi(\|f\|/k)d\mu \leqq 1$ for some $k > 0$. $L^\Phi(\mathfrak{X})$ becomes a normed linear space under either of the two equivalent norms $N_\Phi(f) = \inf\{k > 0: \int_\Omega \Phi(\|f\|/k)d\mu \leqq 1\}$ or, if

[1] The results announced here are contained in the author's doctoral thesis written under the guidance of Professor M. M. Rao at Carnegie Institute of Technology.

μ has the finite subset property, FSP($A \in \Sigma$, $\mu(A) = \infty$, only if there is $B \in \Sigma$, $B \subset A$, $0 < \mu(B) < \infty$), $\|f\|_\Phi = \sup\{\int_\Omega \|f\| \, \|g\| d\mu,\ g \in L^\Psi(\mathfrak{X}^*),\ N_\Psi(g) \leqq 1\}$. $M^\Phi(\mathfrak{X}) \subset L^\Phi(\mathfrak{X})$ is the closed subspace determined by the μ-simple functions. If Φ satisfies the Δ_2 condition ($\Phi(2x) \leqq K^\Phi(x)$), $M^\Phi(\mathfrak{X}) = L^\Phi(\mathfrak{X})$. $L^\Phi(\mathfrak{X})$ and $V^\Phi(\mathfrak{X})$ are related by

THEOREM 1. *Let Φ be continuous, for $f \in L^\Phi(\mathfrak{X})$, define λf by $\lambda f(E) = \int_E f d\mu$, $E \in \Sigma_0$. The mapping λ maps $L^\Phi(\mathfrak{X})$ linearly into $V^\Phi(\mathfrak{X})$ and $N_\Phi(f) = N_\Phi(\lambda f)$. If μ has* FSP *and $f \in M^\Phi(\mathfrak{X})$, $\|f\|_\Phi = \|\lambda f\|_\Phi$.*

2. **The structure of** $V^\Phi(\mathfrak{X})$. When $\Phi(x) = |x|$, the corresponding $V^\Phi(\mathfrak{X})$ is denoted by $V^1(\mathfrak{X})$ and is endowed with the variation norm $\mathfrak{v}(\cdot)$. The study of the structure of $V^\Phi(\mathfrak{X})$ rests upon the following generalization of the Radon-Nikodym-Bochner theorem [4, IV.9].

THEOREM 2. *Let $\mu(\Omega) < \infty$ and $F \in V^1(\mathfrak{X})$. If*

$$\left\{\frac{F(E)}{\mu(E)}: \left\|\frac{F(E)}{\mu(E)}\right\| \leqq n,\ E \in \Phi\right\}$$

is weakly sequentially compact for each positive integer n, then for each $\epsilon > 0$, there exists a μ-simple function f_ϵ such that $\mathfrak{v}(F - \lambda f_\epsilon) < \epsilon$ where λ is the injection of Theorem 1.

For $F \in V^\Phi(\mathfrak{X})$ and each partition $\pi = \{E_n\}$, F_π is defined by

$$F_\pi = \sum_\pi \frac{F(E_n)}{\mu(E_n)} \mu.$$

E_n where $\mu \cdot E_n$ is the set function defined by $\mu \cdot E_n(E) = \mu(E_n \cap E)$, $E \in \Sigma_0$. $S^\Phi(\mathfrak{X})$ denotes the closed subspace of $V^\Phi(\mathfrak{X})$ of functions satisfying $\lim_\pi N_\Phi(F - F_\pi) = 0$ where the limit is taken in the Moore-Smith sense.

THEOREM 3. *If Φ obeys the Δ_2 condition and $\mathfrak{X}$ is reflexive $S^\Phi(\mathfrak{X}) = V^\Phi(\mathfrak{X})$.*

3. **Linear operators on** $V^\Phi(\mathfrak{X})$ **and** $L^\Phi(\mathfrak{X})$. $B(\mathfrak{X}, \mathfrak{Y})$ denotes the B-space of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$.

DEFINITION. $W^\Phi(\Omega, \Sigma, \mu, B(\mathfrak{X}, \mathfrak{Y})) = (W^\Phi(B(\mathfrak{X}, \mathfrak{Y}))$ consists of all finitely additive μ-continuous $B(\mathfrak{X}, \mathfrak{Y})$-valued set functions H defined on Σ_0 and satisfying (i) $y^*H \in V^\Phi(\mathfrak{X}^*)$ for all $y^* \in \mathfrak{Y}^*$ and (ii) $\sup_{\|y^*\| \leq 1} N_\Phi(y^*H) = \|H\|_{W^\Phi} < \infty$.

THEOREM 4. *Let Φ be continuous. Then*

(a) *to each $h \in B(S^\Phi(\mathfrak{X}), \mathfrak{Y})$ there corresponds a unique $H \in W^\Psi(B(\mathfrak{X}, \mathfrak{Y}))$ such that*

$$h(F) = \lim_{\pi} \sum_{\pi} \frac{H(E_n)[F(E_n)]}{\mu(E_n)}, \qquad F \in S^{\Phi}(\mathfrak{X}).$$

(b) *If* $S^{\Phi}(\mathfrak{X})$ *is normed with* $\|\cdot\|_{\Phi}$, *the correspondence* $h \to H$ *maps* $B(S^{\Phi}(\mathfrak{X}), \mathfrak{Y})$ *isometrically isomorphically onto* $W^{\Psi}(B(\mathfrak{X}, \mathfrak{Y}))$.

Since the injection λ of Theorem 1 maps $M^{\Phi}(\mathfrak{X})$ onto a dense subset of $S^{\Phi}(\mathfrak{X})$, it follows that $B(M^{\Phi}(\mathfrak{X}), \mathfrak{Y})$ is equivalent to $W^{\Psi}(B(\mathfrak{X}, \mathfrak{Y}))$ with the representation of $h \in B(M^{\Phi}(\mathfrak{X}), \mathfrak{Y})$ taking the form

$$h(f) = \lim_{\pi} \sum_{\pi} \frac{H(E_n)\left[\int_{E_n} f d\mu\right]}{\mu(E_n)}, \qquad f \in M^{\Phi}(\mathfrak{X}),\ H \in W^{\Psi}(B(\mathfrak{X}, \mathfrak{Y})).$$

4. **Martingales.** Here generalizations of the classical conditional expectation operator and martingales to the $V^{\Phi}(\mathfrak{X})$ setting are given.

DEFINITION. Let Φ obey the Δ_2-condition and B be a subfield of Σ. For $F \in S^{\Phi}(\mathfrak{X})$, $P_B(F)$ is defined by $P_B(F) = \lim_{\pi_B} F_{\pi_B}$ where the limit is taken in the $V^{\Phi}(\mathfrak{X})$ topology through all partitions $\pi_B \subset B$.

P_B and E^B, the usual conditional expectation operator [10] are intimately related. In fact if Σ is a σ-field, B is a sub σ-field of Σ and μ is countably additive and finite on Σ, then $\lambda E^B(f) = P_B(\lambda f)$ for all $f \in L^1(\mathfrak{X})$ where λ is the injection of $L^1(\mathfrak{X})$ into $V^1(\mathfrak{X})$ of Theorem 1.

DEFINITION. Let Φ obey the Δ_2-condition and $\{B_\tau, \tau \in T\}$ be an increasing net of subfields of Σ. $\{F_\tau, B_\tau, \tau \in T\}$ is an $S^{\Phi}(\mathfrak{X})$-martingale if $P_{B_{\tau_1}}(F_{\tau_2}) = F_{\tau_1}$ for $\tau_2 \geqq \tau_1$.

Typical of the class of mean martingale convergence theorems which can be proved is

THEOREM 5. *Let* $\mathfrak{X}$ *be reflexive,* Φ *obey the* Δ_2*-condition and* Ψ *be continuous. If* $\{F_\tau, B_\tau, \tau \in T\}$ *is an* $S^{\Phi}(\mathfrak{X})$*-martingale, then the net* $\{F_\tau, \tau \in T\}$ *converges in* $N_\Phi(\cdot)$ *norm if and only if there exists* P, $0 < P < \infty$ *such that* $N_\Phi(F_\tau) \leqq P$, $\tau \in T$.

The following corollary which extends [3, Theorem 3] is immediate from the properties of λ.

COROLLARY 7. *Let* Σ *be a* σ*-field and* μ *be countably additive and finite on* Σ. *If* Φ *obeys the* Δ_2*-condition and* Ψ *is continuous, a martingale* $\{f_\tau, B_\tau, \tau \in T\}$ *in* $L^{\Phi}(\mathfrak{X})$ *converges in* $L^{\Phi}(\mathfrak{X})$ *norm if and only if there exists* P, $0 < P < \infty$ *such that* $N_\Phi(f_\tau) \leqq P$, $\tau \in T$.

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