SCHOTTKY GROUPS AND LIMITS OF KLEINIAN GROUPS[1]

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DEFINITION. A group of Möbius transformations is said to be a *marked Schottky group* of genus g with *standard generators* $T_1, \cdots, T_g$, if there exist disjoint Jordan curves $\Gamma_1, \Gamma_1', \cdots, \Gamma_g, \Gamma_g'$, which bound a $2g$-times connected domain D, such that $T_j(D) \cap D = \emptyset$ and $T_j(\Gamma_j) = \Gamma_j', j = 1, \cdots, g$.

LEMMA 1. *If G is a Schottky group of genus g, then G is a marked Schottky group on every set of g free generators; i.e., every set of g free generators for G is standard.*

The proof is based on the classical theorem on automorphisms of finitely generated free groups.

LEMMA 2. *Every finitely generated subgroup of a Schottky group is a Schottky group.*

The proof uses the fact that all Schottky groups are quasi-conformally equivalent to certain Fuchsian groups.

REMARK. The preceding two lemmas can be generalized, with appropriate modifications, to *Schottky type groups* (see [1] for the definition).

THEOREM 1. *Let $T_1, \cdots, T_g$ be $g > 1$ Möbius transformations. Suppose there exist marked Schottky groups of genus g, $\langle T_{1,n}, \cdots, T_{g,n} \rangle$, such that* $\lim_{n\to\infty} T_{j,n} = T_j$, $j = 1, \cdots, g$. *Then the group G generated by $T_1, \cdots, T_g$ is a free group on g generators, without elliptic elements.*

The proof uses Lemmas 1 and 2 and involves an elementary area argument.

DEFINITION. An isomorphism $\phi: G_1 \to G_2$ between two Kleinian groups is said to be *type preserving* if $\phi(T)$ is parabolic if and only if T is.

THEOREM 2. *For every $n = 0, 1, 2, \cdots$, let $G(n) = \{T_j(n), j = 0, 1, \cdots\}$ be a Kleinian group. Assume that there are Möbius transformations T_j such that* $\lim_{n\to\infty} T_j(n) = T_j$, *and denote the group $\{T_j, j = 0, 1, \cdots\}$ by G. Assume also that all mappings $T_j(0) \to T_j(n)$*

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of $G(0)$ onto $G(n)$ are type preserving isomorphisms. Then the mapping $T_j(0)\to T_j$ is an isomorphism of $G(0)$ onto G, and G contains no elliptic elements of infinite order.

The proof uses the observation of Myrberg that every Kleinian group contains Schottky subgroups, Theorem 1, and a recent result of Maskit [3].

DEFINITION. We say that two marked Schottky groups $G=\langle T_1, \cdots, T_g\rangle$ and $G'=\langle T_1', \cdots, T_g'\rangle$ are equivalent if there exists a Möbius transformation B such that $BT_iB^{-1}=T_i'$, $i=1, \cdots, g$. The *Schottky space* of genus g, S_g, is the set of all equivalence classes of Schottky groups of genus $g>1$, with the natural topology.

There is a natural homeomorphism of $SL'(2, \mathbf{C})$ ($SL(2, \mathbf{C})$ factored by its center) onto $P_3(\mathbf{C})-X$, where $P_3(\mathbf{C})$ is complex projective 3-space and X the quadric corresponding to singular matrices. Thus the space of groups of Möbius transformations on g generators, $SL'(2, \mathbf{C})^g$, is naturally homeomorphic to $(P_3(\mathbf{C}) - X)^g$. Let $X_0\subset(P_3(\mathbf{C})-X)^g$ be the set of points which, considered as groups of Möbius transformations on g generators, have at least three distinct fixed points for the g generators. Every $B\in SL'(2, \mathbf{C})$ acts on X_0 by sending $G\in X_0$ into $BGB^{-1}\in X_0$. Let Λ be the group of all such transformations. Then X_0/Λ is a complex manifold of dimension $3g-3$ and Schottky space S_g is embedded as an *open connected* submanifold $\hat{S}_g$ in X_0/Λ.

THEOREM 3. *Every point on the boundary of $\hat{S}_g$ represents a free group on g generators, without elliptic elements.*

This follows directly from Theorem 1.

DEFINITION (cf. [2]). A point on the boundary of $\hat{S}_g$ is a *cusp* if the group it represents contains parabolic elements.

THEOREM 4. *A boundary point of $\hat{S}_g$ is either a cusp or represents a free strictly loxodromic group which is not Kleinian, i.e., not discontinuous.*

The proof follows easily from a dimension argument, Theorem 1, and a theorem of Maskit [3].

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REFERENCES

1. L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Univ. Press, Princeton, N. J., 1960.

2. L. Bers and B. Maskit, *On a class of Kleinian groups*, Int. Symposium on Analytic Functions at Erevan (to appear).

3. B. Maskit, *A characterization of Schottky groups*, J. Analyse Math. (to appear).

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