# HOMEOMORPHISMS OF $S^n \times S^1$

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It is the object of this note to describe several results about homeomorphisms of $S^n \times S^1$. The main tool is Theorem 2: Every homeomorphism of $S^n \times S^1$ extends to a homeomorphism of $D^{n+1} \times S^1$. The proof is sketched in §1 and the result used in §2 to yield information about deformations of homeomorphisms. §3 contains results on the division of $S^{n+2}$ by $S^n \times S^1$.

1. DEFINITION. Two submanifolds $L^{n-1}$ and $M^{n-1}$ in $N^n$ are said to be *transverse* if there is a coordinate system about each point of $L^{n-1} \cap M^{n-1}$ in which $L^{n-1}$ and $M^{n-1}$ look like intersecting hyperplanes in $R^n$.

THEOREM 1. *If $\Sigma$ is a locally flat n-sphere in $S^n \times S^1$, $n > 1$, then $\Sigma$ bounds a locally flat $(n+1)$-disk $\Delta$ in $D^{n+1} \times S^1$ which is transverse to $S^n \times S^1$.*

PROOF (SKETCH). If $\Sigma$ bounds a disk in $S^n \times S^1$, the proof is trivial, so assume that it does not. Look at the universal covering space $S^n \times R^1$ of $S^n \times S^1$ with covering translation $T$, and let $\Sigma_0$ be a lifting of $\Sigma$ to $S^n \times R^1$. Since $T$ is stable, the region between $\Sigma_0$ and $T\Sigma_0$ is an annulus (Brown and Gluck [1]). Thus there is a homeomorphism $g: S^n \times R^1 \to S^n \times R^1$ such that $Tg = gT$ and $g(S^n \times \{0\}) = \Sigma_0$. It will be sufficient to construct a disk $\Delta_0 \subset D^{n+1} \times R^1$ such that

(1) $\Delta_0$ is locally flat,

(2) $\Delta_0$ is transverse to $S^n \times R^1$ along $\Sigma_0$, and

(3) $\Delta_0$ is disjoint from its translates $T^k \Delta_0$.

Then $\Delta_0$ will project onto the desired $\Delta$.

CONSTRUCTION OF $\Delta_0$. Choose a number $M$ such that $\Sigma_0 \subset S^n \times (-M, M)$. Let $A$ be the annular region on $S^n \times S^1$ between $\Sigma_0$ and $S^n \times \{M\}$, and $B$ the disk $D^{n+1} \times \{M\}$. Then $A \cup B$ is a locally flat manifold, which is a disk by the generalized Shoenflies theorem (Brown [2], [3]).

Give $R^{n+1}$ polar coordinates $(r, x) \to rx$ where $r \in [0, \infty)$ and $x \in S^n$.

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We push $A\cup B$ inside $D^{n+1}\times R^1$ with a homeomorphism defined in a neighborhood of $A\cup B$ by

$$G(r, x, t) = (r(1 - p_2(g^{-1}(x, t))/2M), x, t),$$

where $p_2(x, t) = t$.

Since $G$ moves a point of $A$ toward the center of $D^{n+1}$ by a distance proportional to its $R^1$ coordinate under $g^{-1}$, $G$ keeps $\Sigma_0$ fixed, and $(1\times g^{-1})G(A)$ lies in a cone $C_0 = \{r, x, t \mid r = 1 - t/2M\}$. Thus if $\Delta_0 = G(A)\cup G(B)$, conditions (1) and (2) are satisfied, and (3) can be checked by verifying that each of the four terms in the expansion of $(G(A)\cup G(B))\cap(T^kG(A)\cup T^kG(B))$ is empty.

THEOREM 2. *Every homeomorphism* $h$: $S^n\times S^1\to S^n\times S^1$, $n>1$, *extends to a homeomorphism* $H$: $D^{n+1}\times S^1\to D^{n+1}\times S^1$.

PROOF. Lift $h$ to a homeomorphism $\tilde{h}$: $S^n\times R^1\to S^n\times R^1$ of the universal covering spaces, and construct $\Delta_0$ as above. Since $\Delta_0$ is a disk, $\tilde{h}\mid S^n\times\{0\}$ extends to a homeomorphism $G_0$: $D^{n+1}\times\{0\}\to\Delta_0$. Define an embedding $G_1$: $\partial(D^{n+1}\times[0, 1])\to R^{n+1}\times R^1$ by $G_1\mid D^{n+1}\times\{0\} = G_0$, $G_1\mid D^{n+1}\times\{1\} = TG_0T^{-1}$, and $G_1\mid S^n\times[0, 1] = \tilde{h}\mid S^n\times[0, 1]$. Since $\Delta_0$ is transverse to $S^n\times R^1$, the sphere $G_1(\partial(D^{n+1}\times[0, 1])) = \Delta_0\cup T\Delta_0 \cup\tilde{h}(S^n\times[0, 1])$ is locally flat. Thus, by the generalized Shoenflies theorem, $G_1$ extends to an embedding $G$: $D^{n+1}\times[0, 1]\to D^{n+1}\times R^1$. Since $GT = TG$ whenever both sides are defined, $G$ projects to the desired homeomorphism $H$.

REMARK. If $h$ is piecewise linear and $n+1>4$, then $H$ can be made piecewise linear by the Hauptvermutung for cells. Also, if $n=1$, the above proof is valid whenever the lifting $\tilde{h}$: $S^1\times R^1\to S^1\times R^1$ exists, as is the case for Theorem 3.

COROLLARY. *Let* $M^{n+2}$ *be a manifold constructed by identifying* $S^n\times D^2$ *and* $D^{n+1}\times S^1$ *using some homeomorphism* $h$: $S^n\times S^1\to S^n\times S^1$ *of their boundaries. Then* $M^{n+2}$ *is homeomorphic to* $S^{n+2}$ *for* $n>1$.

2. DEFINITION. Let $h_0$ and $h_1$ be homeomorphisms of $M$ onto itself. A homeomorphism $H$: $M\times[0, 1]\to M\times[0, 1]$ is called a *weak isotopy* (or *concordance*) between $h_0$ and $h_1$ if $H(x, 0) = (h_0(x), 0)$ and $H(x, 1) = (h_1(x), 1)$.

THEOREM 3. *Let* $H$ *be a homeomorphism of* $D^n\times S^1$, $h$ *its restriction to* $S^{n-1}\times S^1$, *and* $g$ *a weak isotopy between* $h$ *and the identity. Then* $g$ *extends to a weak isotopy* $G$ *between* $H$ *and the identity.*

PROOF. Consider $D^n\times S^1\times[0, 1]$ as $D^{n+1}\times S^1$. Then the desired map $G$ has already been defined on $S^n\times S^1$, so apply Theorem 2.

COROLLARY. *Let* WI$(X)$ *be the group of weak isotopy classes of homeomorphisms of* $X$. *Then the inclusion map* $i$: $S^n \times S^1 \to D^{n+1} \times S^1$ *induces an isomorphism*

$$i^*: \mathrm{WI}(D^{n+1} \times S^1) \to \mathrm{WI}(S^n \times S^1) \quad \textit{for } n > 1.$$

CONJECTURE 1. $\mathrm{WI}(S^n \times S^1) = Z_2 + Z_2 + Z_2$ for $n \geqq 2$. By obstruction theory, the group of homotopy equivalences of $S^n \times S^1$ with itself is $Z_2 + Z_2 + Z_2$. Thus the conjecture is that every homeomorphism of $S^n \times S^1$ which is homotopic to the identity is weakly isotopic to the identity. The following theorems are partial results in this direction.

THEOREM 4. *Every homeomorphism* $h$ *of* $S^n \times S^1$, $n \geqq 2$, *is weakly isotopic to a homeomorphism* $h'$ *such that*

$$h'(S^n \times \{0\}) \subset S^n \times (-\epsilon, \epsilon), \quad \textit{for any given } \epsilon > 0.$$

THEOREM 5. *Every stable homeomorphism of* $S^n \times S^1$, $n \geqq 2$, *which is homotopic to the identity is weakly isotopic to one which is fixed on* $S^n \times \{0\} \cup \{0\} \times S^1$.

THEOREM 6. *Every homeomorphism* $H$ *of* $S^n \times S^1$ *which is piecewise linear in a neighborhood of* $\{0\} \times S^1$ *and homotopic to the identity is weakly isotopic to the identity.* (*Cf. Browder* [6]).

PROOF. We may assume the neighborhood is $D^n \times S^1$ with boundary $S^{n-1} \times S^1$. The cases $n = 1$ and $n = 2$ follow from Gluck [4]. If $n \geqq 3$, unknot $H(\{0\} \times S^1)$ piecewise linearly, by Guggenheim [9]. Then by the regular neighborhood theorem there is a piecewise linear isotopy which moves $H(D^n \times S^1)$ onto $D^n \times S^1$. By induction, $H | S^{n-1} \times X^1$ is weakly isotopic to the identity. Then apply Theorem 3.

3. The following two theorems are related to the unknotting of $S^n \times S^1$ in $S^{n+2}$ (cf. Goldstein [7]). Let $f$: $S^n \times S^1 \to S^{n+2}$ be a locally flat embedding, and let $A_1$ and $A_2$ be the closures of the components of $S^{n+2} - f(S^n \times S^1)$. By the Mayer-Vietoris theorem, one of them, say $A_1$, has the homology of $S^1$, and the other, $A_2$, has the homology of $S^n$.

THEOREM 7. *If* $f$ *is also piecewise linearly locally flat in a neighborhood of* $S^1$, *and* $n \geqq 3$, *then* $A_2$ *is homeomorphic to* $S^n \times D^2$.

PROOF (SKETCH). By the van Kampen theorem, $A_2$ is simply connected. By general position, embed a piecewise linear 2-disk $D$ in $A_2$ with $\partial D = f(\{0\} \times S^1)$. Let $N$ be the closed star of $D$ in a suitable triangulation of $A_2$ near $D$. Since $A_2$ is a combinatorial manifold near $D$, $N$ is a ball by the regular neighborhood theorem. By the corollary to Theorem 2 and the Shoenflies theorem, the closure of $A_2 - N$ is also a ball, and one can check that the two balls fit together to make $S^n \times D^2$.

REMARK. If Theorem 7 were true without the extra assumptions, Conjecture 1 for stable homeomorphisms would follow from Theorem 5.

THEOREM 8. *If $f(S^n \times \{0\})$ is unknotted in $S^{n+2}$, then some finite $k$-fold covering space of $A_1$ is homeomorphic to $D^{n+1} \times S^1$, for $n \geqq 3$.*

PROOF. Let $A$ be the two point compactification of the universal covering space $\tilde{A}_1$ of $A_1$. The boundary of $A$ is locally flat except possibly at the two added points. Therefore, by Hutchinson [8], $A$ is a ball, and $\tilde{A}_1$ is homeomorphic to $D^{n+1} \times R^1$. Let $\tilde{f}(S^n \times \{0\})$ span a nice disk $\Delta_0$ in $\tilde{A}_1$, and choose a large $k$ so that $\Delta_0$ is disjoint from $T^k\Delta_0$. Then proceed as in Theorem 2.

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