SOME THEOREMS ON FACTORIZATION OF MEROMORPHIC FUNCTIONS

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In [3] the author proved

THEOREM 1.[1] *If f is any entire function of lower order less than $\frac{1}{2}$ and g is entire, then $f(g)$ is periodic if and only if g is.*

By means of a result due to Edrei [1] and Ostrovskii [6] it is possible to generalize Theorem 1 to a certain class of meromorphic functions. We begin with

LEMMA 1 (EDREI [1], OSTROVSKII [6]). *Let $f(z)$ be meromorphic of lower order $\lambda < \frac{1}{2}$. If $\delta(\infty, f) > 1\text{-}\cos \pi\lambda$, then $|f(re^{i\theta})| \to \infty$, uniformly in θ as $r_n \to \infty$ through a suitable sequence.*

Here δ is the Nevanlinna deficiency (see Hayman [5, p. 42]).

THEOREM 2. *Let f be meromorphic of lower order λ and let g be entire. If $0 \leqq \lambda < \frac{1}{2}$ and for some a, $\delta(a, f) > 1 - \cos \pi\lambda$, then $f(g)$ is periodic if and only if g is. If τ is a period of $f(g)$, then g has a period having the same argument as τ.*

SKETCH OF PROOF. We assume that $f(g)$ is periodic with period τ having argument α. Let L be the half line $re^{i\alpha}$ everywhere except near poles of $f(g)$, where we let L loop around them with radius ϵ, ϵ a small positive number. Letting $f^*(z) = 1/(f(z) - a)$ and applying Lemma 1 we see that $|f^*(re^{i\theta})| \to \infty$, uniformly in θ as $r_n \to \infty$ through a suitable sequence. From the hypotheses of the theorem it follows that $f(g)$ is bounded on L. If g is bounded on L, then as in the proof of Theorem 1 (see [3]) g must be periodic with a period having the same argument as τ. If g is unbounded on L, then f is bounded on $g(L)$ and this leads to a contradiction via Lemma 1.

COROLLARY. *If P is a polynomial and f is as in Theorem 2, then $f(P)$ is not periodic.*

This Corollary is a partial solution to the more general question suggested in [4]: If f is meromorphic for which polynomials is $f(P)$ periodic?

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[1] N. Baker proved an analogue of this theorem for f of order $<1/2$. See *On some results of A. Renyi and C. Renyi concerning periodic entire functions*, Acta Sci. Math. (Szeged) **27** (1966), 197–200.

Theorem 2 also yields a generalization of an earlier result mentioned in [4].

THEOREM 3. *Let f be meromorphic and g entire. If $f(g)$ is of finite order, has no deficient values and is periodic, with period τ, then either f has no deficient values or g is periodic with a period having the same argument as τ.*

SKETCH OF PROOF. By a theorem of Edrei and Fuchs [2] either f is of zero order or g is a polynomial. In the latter case f can certainly not have any deficient values since $f(g)$ does not. In the former case one can apply Theorem 2 and arrive at the desired conclusion.

COROLLARY (SEE [4]). *Let f be meromorphic and g entire. If $f(g)$ is elliptic, then f has no deficient values.*

This last corollary rules out the possibility of applying the earlier one to resolve the question mentioned in [4]: If P is a polynomial of degree n, where $n=5$ or $n \geqq 7$ and f is any meromorphic function, then $f(g)$ is not elliptic?

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