THE OBSTRUCTION TO AN AUTOMORPHISM OF A FILTERED RING

MURRAY GERSTENHABER[1]

Communicated January 31, 1968

This paper sketches the proofs that (1) an automorphism of a complete filtered ring is a limit of successive approximations, (2) given an nth order approximate automorphism, there is an obstruction to prolonging it to an $(n+1)$st order approximation, the obstruction lying in a certain 2nd cohomology group, and (3) the mapping which sends an nth order approximate automorphism to its obstruction is a crossed homomorphism from the multiplicative group of nth order approximate automorphisms to the (additive) 2nd cohomology group containing the obstructions. The rings in question need not be associative: we tacitly assume that there is given a "category of interest," $\mathcal{C}$, in the sense of [1] (which may be, in particular, the category of associative, Lie, or commutative associative rings), and "ring" and "morphism" are meant relatively to $\mathcal{C}$. The cohomology groups are the Yoneda-type groups introduced in [1], but note that for the categories of associative, Lie, and commutative associative algebras over a fixed coefficient field these coincide, respectively, with the Hochschild, Chevalley-Eilenberg, and Harrison groups (cf. [1] and [2]).

1. Recall, following [1], that a complete filtered ring $A\supset \cdots \supset F_iA\supset F_{i+1}A\supset \cdots$ is itself a limit of successive approximations. For let $A((t))$ be the ring of formal power series $\sum_{i=-n}^{\infty} a_it^i$, $a_i\in A$, let $\mathrm{App}\,A$ be the subring of those power series with $a_i\in F_iA$, let $F_j(\mathrm{App}\,A)=\{\sum a_it^i \mid a_i\in F_{i+j}A\}=t^{-j}\mathrm{App}\,A$, and set $\mathrm{App}_j\,A=\mathrm{App}\,A/F_{j+1}\mathrm{App}\,A$. Then $\mathrm{App}_0\,A$ is the completion of the associated graded ring of A, there are natural epimorphisms $\mathrm{App}_0\,A\leftarrow\mathrm{App}_1\,A\leftarrow\mathrm{App}_2\,A\leftarrow \cdots$, $\mathrm{App}\,A$ is the inverse limit of this sequence, and letting $(t^{-1}-1)$ denote the ideal of $\mathrm{App}\,A$ consisting of all $t^{-1}\alpha-\alpha$, $\alpha\in\mathrm{App}\,A$, there is a natural isomorphism $\mathrm{App}\,A/(t^{-1}-1)\cong A$. Thus A can be recaptured from the successive approximations $\mathrm{App}_n\,A$. For simplicity, we henceforth denote $\mathrm{App}_n\,A$ by A_n and $\mathrm{App}\,A$ by A_∞. The latter has a gradation in which the homogeneous elements of degree i are of the form at^i, $a\in F_iA$. This gradation is compatible with the filtration, and induces a gradation on

[1] The author gratefully acknowledges the support of the Institute for Advanced Study and of the N.S.F. through grant GP 3683 with the University of Pennsylvania.

every A_n. Further, A_∞ has an additive endomorphism $\alpha \to t^{-1}\alpha$ which reduces gradation and increases filtration; it will be denoted simply t^{-1}, as will the endomorphism which it induces on every A_n.

A filtration-preserving morphism $f: A \to B$ of complete filtered rings induces for every n (including ∞) a morphism $f_n: A_n \to B_n$ which respects the gradation and commutes with t^{-1}, and f_∞ is the inverse limit of the f_n. Since f_∞ carries the ideal $(t^{-1}-1)$ of A_∞ into the corresponding ideal of B, it induces a morphism of $A_\infty/(t^{-1}-1) = A$ into $B_\infty/(t^{-1}-1) = B$, and this induced morphism is just f itself. Thus f may be viewed as the limit of the approximations f_n. Note moreover that any gradation-preserving morphism $A_\infty \to B_\infty$ which commutes with t^{-1} is of the form f_∞ for some filtration preserving $f: A \to B$.

There are other ways to approximate a filtered ring A, for example by using Trunc A, the subring of App A consisting of those elements which are just polynomials in t^{-1} (cf. Rim [5], following Guillemin-Sternberg [4]).

2. If M is a module over a (not necessarily filtered) ring A then $\mathcal{E}^2(A, M)$ will denote the Baer group of equivalence classes of singular extensions $0 \to M \to B \to A \to 0$. When A and M are graded then it will be tacitly understood that so is B and that all the morphisms are of degree 0. The group $\mathcal{E}^3(A, M)$ consists, following [1], of classes of "admissible sequences" $\boldsymbol{E}$: $0 \to M \to N \xrightarrow{\rho} B \to A \to 0$. In the associative, Lie, and commutative associative cases, these are exact sequences of rings (M being a zero ring) and morphisms in which B operates on N, where $\rho(nn') = n\rho(n') = \rho(n'n)$ for all $n, n' \in N$, and where, letting B operate on M by means of the epimorphism $B \to A$, the morphism $M \to N$ respects the operation of B. The sequence $\boldsymbol{E}$ represents 0 if and only if there is a "solution" (commutative diagram)

$$\begin{array}{ccccccccc} & & 0 & \to & N & \xrightarrow{\lambda} & \overline{B} & \to & A & \to & 0 \\ & & & & \| & & \mu\downarrow & & \| & & \\ 0 & \to & M & \to & N & \to & B & \to & A & \to & 0, \end{array}$$

in which case the set of solutions forms, in a natural way, a principal homogeneous space over $\mathcal{E}^2(A, M)$.

Now, A being filtered and complete, let us define on the underlying additive group of A_n a new multiplication sending (α, β) to $t^{-i}\alpha\beta$, and denote this new ring by $t^{-i}A_n$. Then for every nonnegative m, n, and $k \leqq n$ there is a monomorphism i: $t^{-m-k} A_{n-k} \to t^{-m}A_n$, an epimorphism π: $t^{-m}A_n \to t^{-m}A_{n-k}$, and every ring morphism $f: A_m \to A_n$ which commutes with t^{-1} induces a morphism $t^{-i}A_m \to t^{-i}A_n$. The latter will still be denoted by f. There is, for every $n \geqq 0$, an admissible sequence

$$E: 0 \to t^{-n-1}A_0 \to t^{-1}A_n \to A_n \to A_0 \to 0.$$

This represents 0 since it has a "trivial" solution in which $\bar{B}$ is A_{n+1}, λ is $i: t^{-1}A_n \to A_{n+1}$ and μ is $\pi: A_{n+1} \to A_n$. Note that all the ring morphisms considered here preserve filtration, gradation, and commute with t^{-1}; this will always be tacitly assumed.

An nth order approximate automorphism of A is by definition an automorphism of A_n; these form a group, $\operatorname{Aut}_n A$. When $f \in \operatorname{Aut}_n A$ is given there is another solution for E in which $\bar{B}$ is still A_{n+1}, but in which $\lambda = if^{-1}$ and $\mu = f\pi$. Denoting by e_f the class of this solution and by e_0 that of the trivial one, the element $\operatorname{Obs} f = e_f - e_0$ of $\mathcal{E}^2(A_0, t^{-n-1}A_0)$ vanishes if and only if f can be extended to an automorphism of A_{n+1}. For that, in effect, is what it means for the two solutions to be equivalent.

3. If $f, g \in \operatorname{Aut}_n A$, then

$$\operatorname{Obs} fg = e_{fg} - e_0 = (e_{fg} - e_f) + (e_f - e_0).$$

The second term is just $\operatorname{Obs} f$. Now $\operatorname{Aut}_0 A$ operates in a natural way on $\mathcal{E}^2(A_0, t^{-n-1}A_0)$ and there is a natural morphism $\operatorname{Aut}_n A \to \operatorname{Aut}_0 A$ by means of which $\operatorname{Aut}_n A$ also operates. It is easy to verify that $e_{fg} - e_f = f(e_g - e_0) = f \operatorname{Obs} g$, yielding

THEOREM 1. $\operatorname{Obs} fg = f \operatorname{Obs} g + \operatorname{Obs} f$.

Denoting by $F_1 \operatorname{Aut}_n A$ *the kernel of the morphism* $\operatorname{Aut}_n A \to \operatorname{Aut}_0 A$, *it follows that* $\operatorname{Obs} | F_1 \operatorname{Aut}_n A \to \mathcal{E}^2(A_0, t^{-n-1}A_0)$ *is a group morphism.*

This is the general statement of the "Obstruction Morphism Theorem" of [3].

REFERENCES

1. M. Gerstenhaber, *On the deformation of rings and algebras*. II, Ann. of Math. **84** (1966), 1–19.

2. ———, *The third cohomology group of a ring and the commutative cohomology theory*, Bull Amer. Math. Soc. **73** (1967), 950–954.

3. ———, *On the deformation of rings and algebras*. III, Ann. of Math., (to appear).

4. V. Guillemin and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964), 16–47.

5. D. S. Rim, *Deformation of transitive Lie algebras*, Ann. of Math. **83** (1966), 339–357.

UNIVERSITY OF PENNSYLVANIA