REMARKS ON WEAK TYPE INEQUALITIES FOR OPERATORS COMMUTING WITH TRANSLATIONS

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The purpose of this note is to show that by very simple arguments one can obtain an analogue of E. M. Stein's theorem [1] for noncompact σ-compact groups. Together with Theorem 1 of Stein we get the following

THEOREM I. *Let G be a locally compact σ-compact group. T_m a sequence of bounded linear operators of $L^p(G)$ into itself $1 \leqq p \leqq 2$ such that*

(a) $T_m(f_g)(x) = T_m(f)(gx)$ *where* $f(gx) = f_g(x)$.

(b) *The support of $T_m(f)$ is contained in a compact set whenever f has compact support.*

Define $M(f)(x) = \sup_m |T_m(f)(x)|$. *Then the following conditions are equivalent*:

1°. $\forall f \in L^p(G)$ $M(f)(x) < \infty$ *a.e. and* $|\{x\colon M(f)(x) > \lambda\}| < \infty$ *for some λ (depending on f).*

2°. $|\{x\colon M(f)(x) > \lambda\}| \leqq C(\|f\|_p/\lambda)^p$ *for* $\lambda \geqq \|f\|_p$ *where C is independent of f.*

Here $|E|$ denotes the left Haar measure of the set E.

Moreover, if G is not compact, the restriction $p \leqq 2$ is not necessary.

It is evident that Theorem I reduces to Theorem 1 in [1] when G is compact. We are going to consider the case G noncompact. As will be seen from an example the restriction $\|f\|_p \leqq \lambda$ is necessary. However, if we replace condition (a) by condition 3° of Theorem II below we get a global weak type estimate.

DEFINITION. A is an affine map on G if it can be represented as a composition of left and right translations with continuous automorphisms of G.

From the uniqueness of left Haar measure

$$\Delta_A \int_G f_A(x)dx = \int_G f(x)dx \quad \text{where} \quad f_A = f(Ax)$$

for some constant $\Delta_A > 0$.

THEOREM II. *Let M be a sublinear operator*[1] *defined on $L^p(G)$,*

[1] M is sublinear if and only if $|M(f+g)(x)| \leqq |M(f)(x)| + |M(g)(x)|$ and $|M(\lambda f)(x)| = |\lambda||M(f)(x)|$.

$1 \leqq p \leqq \infty$, *into measurable functions such that*

1°. $\forall f \in L^p, M(f)(x) < \infty$ *a. e. and* $|\{x : M(f)(x) > \lambda\}| < \infty$ *for some* λ.

2°. *If* $f_k \to f$ *in norm in* L^p *then there exists a subsequence* f_{i_k} *such that*

$$M(f)(x) \leqq \liminf_{k \to \infty} M(f_{i_k})(x) \quad a.e.$$

3°. *There exists an affine map* A *with* $\Delta_A \neq 1$ *such that*

$$M(f_A)(x) = \Delta_A^{\alpha} M(f)(Ax) \quad \textit{for some } \alpha \in \mathbf{R}$$

then

$$|\{x : M(f)(x) > \lambda\}| \leqq C(\|f\|_p/\lambda)^q,$$

where $1/q = 1/p + \alpha$.

Remark. The condition $\Delta_A \neq 1$ implies already that G cannot be compact. We will prove later that condition 2° is verified for M in Theorem I.

The proof is based on the following lemma of Edwards and Hewitt [2].

Lemma (Edwards-Hewitt [2, Theorem (1, 5)]). *Let* M *satisfy conditions* 1° *and* 2° *of Theorem* II. *Then*

$$\text{(1.1)} \qquad \begin{aligned} &\forall f \in L^p, \quad \forall \lambda > 0, \quad \textit{there exists } C(\lambda) \textit{ such that} \\ &|\{x : M(f)(x) > C(\lambda)\|f\|_p\}| \leqq \lambda. \end{aligned}$$

Proof. Condition 1° is clearly equivalent to the following:

1°′. $\forall f \in L^p$, $|\{x : M(f)(x) > \lambda\}| \to 0$, $\lambda \to \infty$.

Let $E_{n,\lambda} = \{f \in L^p : |\{x : M(f)(x) > n\}| \leqq \lambda\}$, then 1°′ implies $\bigcup_{n=0}^{\infty} E_{n,\lambda} = L^p$. Moreover, $E_{n,\lambda}$ are closed (this follows immediately form 2°). Applying Baire's Category Theorem there exists $E_{n_\lambda,\lambda}$ containing a ball S_λ of radius r_λ centered at f. Clearly every element of the ball in L^p of radius r_λ centered at 0 is a difference of 2 functions in $E_{n_\lambda,\lambda}$ so that (1.1) holds for some constant C_λ.

Proof of Theorem II. Let

$$C(\lambda) = \inf\{C_\lambda \geqq 0 \colon \forall f \in L^p, |\{x : M(f)(x) > C_\lambda \|f\|_p\}| \leqq \lambda\},$$

then

$$|\{x : M(f_A)(x) > C(\lambda)\|f_A\|_p\}| \leqq \lambda$$

but $\|f_A\|_p = \Delta_A^{-1/p}\|f\|_p$ and condition 3° imply

$$\Delta^{-1}\left|\{x : M(f)(x) > C(\lambda)\Delta^{-(\alpha+1/p)}\|f\|_p\}\right| = \left|\{x : M(f)(Ax) > C(\lambda)\Delta^{-(\alpha+1/p)}\|f\|_p\}\right| = \left|\{x : M(f_A)(x) > C(\lambda)\|f_A\|_p\}\right| \leqq \lambda$$

thus

$$C(\lambda\Delta) \leqq C(\lambda)\Delta^{-(\alpha+1/p)}.$$

Clearly 3° holds for A replaced by A^k (k integer) and Δ by Δ^k so that

$$C(\lambda\Delta^k) \leqq C(\lambda)(\Delta^k)^{-(\alpha+1/p)}.$$

Since $C(\lambda)$ is decreasing we get $C(\lambda) \leqq C\lambda^{-(\alpha+1/p)}$. Q.E.D.

PROOF OF THEOREM I. Let us prove first that condition 2° of Theorem II is verified. Let $f_k \to f$ in L^p; since T_n is continuous in measure, we can extract a subsequence such that $T_n(f_{k_i})(x) \to T_n(f)(x)$ a.e.; by a diagonalization process we can choose the subsequence independent of n. Assume now that

$M(f)(x) > \lambda$; there exists $n(x)$ such that $T_{n(x)}(f)(x) > \lambda$, so there exists $k(x)$ such that

$$k > k(x) \Rightarrow T_{n(x)}(f_k)(x) > \lambda,$$

which implies

$$M(f_k)(x) > \lambda,$$

hence $\liminf_{k\to\infty} M(f)(x) \geqq \lambda$.

Using the lemma we get

$$\left|\{x : M(f)(x) > C(\lambda)\|f\|_p\}\right| \leqq \lambda.$$

Our purpose is to compute $C(\lambda)$. Define

$$M_N(f)(x) = \sup_{n \leqq N} \left| T_n(f)(x) \right|$$

then

$$\left|\{x : M_N(f)(x) > C(\lambda)\|f\|_p\}\right| \leqq \left|\{x : M(f)(x) > C(\lambda)\|f\|_p\}\right| \leqq \lambda$$

and

$$\left|\{x : M_N(f)(x) > C(\lambda)\|f\|_p\}\right| \to \left|\{x : M(f)(x) > C(\lambda)\|f\|_p\}\right|, \quad N \to \infty.$$

We now need the following simple lemma.

LEMMA. *Let G be a locally compact noncompact group. Let K be a compact subset, then there exists $h \in G$ such that $hK \cap K = \varnothing$.*

PROOF. Let L be the union of all hK for which $hK\cap K\neq\emptyset$. Clearly $L\subseteq KK^{-1}K$, which is compact. If the lemma is false then $L=G$ compact, contrary to our assumption.

Let $f\in L^p$ have compact support. By the lemma, there exists $h_N\in G$ such that supp $f\cap$supp $f_{h_N}=\emptyset$ and supp $T_mf\cap$supp $T_m(f)_{h_N}=\emptyset$ for all $m\leqq N$. Thus

$$M_N(f+f_{h_N})=M_N(f)+M_N(f)_{h_N},\text{ and supp } M_N(f)\cap\text{ supp } M_N(f_{h_N})=\emptyset.$$

$$\begin{aligned}\lambda &\geqq \left|\left\{x: M_N(f+f_{h_N})(x) > \|f+f_{h_N}\|_pC(\lambda)\right\}\right| \\ &= \left|\left\{x: M_N(f)(x)+M_N(f)(h_Nx) > 2^{1/p}\|f\|_pC(\lambda)\right\}\right| \\ &= 2\left|\left\{x: M_N(f)(x) > 2^{1/p}C(\lambda)\|f\|_p\right\}\right|.\end{aligned}$$

Repeating the argument for k translates, we get

$$\left|\left\{x: M(f)(x) > k^{1/p}C(\lambda)\|f\|_p\right\}\right| \leqq \lambda/k, \qquad k>0 \text{ integer}$$

for all f with compact support. Using 2° of Theorem II, we extend the inequality for all $f\in L^p$ and the theorem follows.

REMARK. We proved Theorem I under the assumption T_m are continuous in measure.

EXAMPLE. Let

$$M(f)(x)=\int_{R^n}\frac{f(x-y)}{|y|^\alpha}\,dy,\qquad 0<\alpha<n,\ f\in L^p.$$

Then

$$M(f)(x)=\int_{|y|\leq 1}\frac{f(x-y)}{|y|^\alpha}\,dy+\int_{|y|>1}\frac{f(x-y)}{|y|^\alpha}\,dy.$$

The first integral is a convolution with an integrable function so it belongs to L^p. The second is a convolution with an L^{p_1} function for $p_1>n/\alpha$ so it belongs to L^{q_1}, $1/q_1=1/p+1/p_1-1$. Thus condition 1° of Theorem II is verified. Moreover, it is clear that

$$M(f_\lambda)(x)=\int_{R^n}\frac{f(\lambda x)}{|x-y|^\alpha}\,dy=\lambda^{-n+\alpha}M(f)(\lambda x)$$

so that condition 3° of Theorem II is verified (the affine map is a dilation). By Theorem I, M maps L^p into weak L^q continuously for $1/q=1/p+\alpha/n-1$.

We can easily improve the result to get a theorem of Stein and Weiss [3]. Consider

$$M(f)(x)=\frac{1}{|x|^\gamma}\int_{R^n}\frac{f(y)}{|x-y|^\beta|y|^\alpha}\,dy$$

where $f \in L^p$, $0<\beta<n$, $\alpha<n(1-1/p)$, $\alpha+\gamma \geqq 0$, $1/p-1+(\alpha+\beta+\gamma)/n > 0$. Then

$$M(f_\lambda)(x) = \frac{1}{|x|^\gamma} \int_{R^n} \frac{f(\lambda y)}{|x-y|^\beta |y|^\alpha} dy = \lambda^{-n+\alpha+\beta+\gamma} M(f)(\lambda x).$$

So that M will map L^p into weak L^q, $1/q = 1/p + (\alpha+\beta+\gamma)/n - 1$ if we can prove 1° Theorem II. But

$$M(f)(x) = \frac{1}{|x|^\gamma} \int_{|y|<|x|/2} \frac{f(y)}{|x-y|^\beta |y|^\alpha} dy + \frac{1}{|x|^\alpha} \int_{|x|/2 \leqq y} \frac{f(y)}{|x-y|^\beta |y|^\alpha} dy.$$

Now if $|y| < \frac{1}{2}|x|$ then $|x-y| > \frac{1}{2}|x|$ so that

$$M(f)(x) \leqq \frac{C}{|x|^{\beta+\gamma}} \int_{|y|<|x|/2} \frac{|f(y)|}{|y|^\alpha} dy + \frac{1}{|x|^{\alpha+\gamma}} \int_{R^n} \frac{|f(y)|}{|x-y|^\beta} dy.$$

Applying Holder's inequality to the first integral for $\alpha < n(1-1/p)$, we get

$$M(f)(x) \leqq \frac{C\|f\|_p}{|x|^{\alpha+\beta+\gamma-n(1-1/p)}} + \frac{1}{|x|^{\alpha+\gamma}} \int \frac{|f(y)|}{|x-y|^\beta} dy.$$

It is clear, using the previous example and conditions on α, β, γ, that 1° is verified.

REMARK. To see that the restriction $\|f\|_p \leqq \lambda$ in Theorem I is essential let us consider

$$T_n(f)(x) = \int_{1/n<|y|<n} \frac{f(x-y)}{|y|^\alpha} dy.$$

Then T_n maps L^p continuously into itself, $M(f)$, however, will map L^p into weak L^q, $1/q = 1/p + \alpha/n - 1$, i.e. $q > p$, and by the proof of Theorem II, will be identically 0 if the inequality of Theorem I were to hold without the restriction $\|f\|_p \leqq \lambda$.

REFERENCES

1. E. M. Stein, *On limits of sequences of operators*, Ann. of Math. **74** (1961), 140–170.

2. R. E. Edwards and E. Hewitt, *Pointwise limits for sequences of convolution operators*, Acta Math. **113** (1965), 181–218.

3. E. M. Stein and Guido Weiss, *Fractional integrals on n-dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514.

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