STRONG CARLEMAN OPERATORS ARE OF HILBERT-SCHMIDT TYPE

JOACHIM WEIDMANN

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This is the solution of a problem posed by G. Targonski [6, §13 V]: Do unbounded strong Carleman operators exist? In fact, we shall prove that every strong Carleman operator is a Hilbert-Schmidt operator (hence it is certainly bounded).

1. Definitions and known results. Carleman operators are usually defined in the space $L_2(a, b)$ where $a \geqq -\infty$, $b \leqq \infty$; without any restriction of generality we may assume that $-\infty < a < b < \infty$. There are several definitions of a Carleman operator used in the literature (e.g. T. Carleman [1], M. Stone [5], G. Targonski [6]; for "semi-Carleman operators" see M. Schreiber [4]). We shall mainly follow the definition used by G. Targonski, but in addition to his definition we shall assume that a Carleman operator is densely defined (e.g. §3).

DEFINITIONS. A densely defined operator K in the Hilbert space $L_2(a, b)$ is called a *Carlemann operator* if it allows a representation of the form

$$(Kf)(x) = \int_a^b K(x, y)f(y)dy \quad \text{for almost all } x,$$

where $\int_a^b |K(x, y)|^2 dy < \infty$ for almost all x. The domain of K consists of all elements $f \in L_2(a, b)$ such that $\int_a^b K(x, y)f(y)dy$ (which is defined for almost all x) represents an element of $L_2(a, b)$.

An operator K is called a *strong Carleman operator* if UKU^* is a Carleman operator for every unitary operator U. An operator K in a Hilbert space is a *Hilbert-Schmidt operator* (or K is of Hilbert-Schmidt type) if for every orthonormal system (ϕ_n), $\sum_n |K\phi_n|^2 < \infty$ (this series has the same value for all complete orthonormal systems).

It is known (e.g. [6]) that every Hilbert-Schmidt operator is a strong Carleman operator. In [6] it is also shown that bounded strong Carleman operators are of Hilbert-Schmidt type. Using the result of this note we may say: *An operator in $L_2(a, b)$ is a strong Carleman operator if and only if it is a Hilbert-Schmidt operator.*

We shall use the following known results:

THEOREM I ([6, LEMMATA 9.1 AND 9.2]). *If K is a strong Carleman operator and B is bounded, then BK and KB are strong Carleman operators.*

THEOREM II ([3, SATZ 4], [6, PROOF OF LEMMA 9.5]). *For every selfadjoint Carleman operator,* 0 *is a limit point of its spectrum; the spectrum of a selfadjoint strong Carleman operator has at most the limit points* $-\infty$, 0 *and* ∞.

THEOREM III ([2, VI.2.7]). *A densely defined closed operator* K *in a Hilbert space can be factorized as* $K=UT$, *where* T *is selfadjoint (nonnegative) and* U *is a partial isometry with initial set* $\mathrm{Cl}(\boldsymbol{R}(T))$ *and final set* $\mathrm{Cl}(\boldsymbol{R}(K))$ ($\mathrm{Cl}=$*closure*).

2. **Proofs.** The proof of our first theorem is almost the same as the proof of [6, Theorem 9.2].

THEOREM 1. *Every selfadjoint strong Carleman operator is of Hilbert-Schmidt type.*

PROOF. Let K be a selfadjoint strong Carleman operator.Theorem II asserts that the spectrum of K consists of a sequence (λ_n) of eigenvalues with limit point 0 (and eventually $\pm\infty$). Since K is a strong Carleman operator there exists for any complete orthonormal system (ϕ_n) a unitary transformation U and a kernel $K_U(x, y)$ such that

$$\int_a^b |K_U(x, y)|^2 dy < \infty,$$

$$(UKU^*\rho_n)(x) = \lambda_n\rho_n(x), \quad (UKU^*f)(x) = \int_a^b K_U(x, y)f(y)dy$$

for almost all x and $f\in\boldsymbol{D}(UKU^*)=U\boldsymbol{D}(K)$. This implies that $\lambda_n\rho_n(x)$ are the Fourier coefficients of the $\boldsymbol{L}_2$-function $K_U(x, y)$ (as a function of y) with respect to the complete orthonormal system (ρ_n). Since $K_U(x, y)$ (as a function of y) is in $\boldsymbol{L}_2(a, b)$ for almost all x, this implies $\sum_n |\lambda_n\rho_n(x)|^2 < \infty$ for almost all x. Let us now chose the complete orthonormal system

$$\rho_n(x) = (b-a)^{-1/2} \exp\{2\pi inx/(b-a)\};$$

it follows that $\sum_n|\lambda_n|^2<\infty$, i.e. K is a Hilbert-Schmidt operator.

THEOREM 2. *Every Carleman operator is closed.*

PROOF. Let K be a Carleman operator, $(u_n)\subset\boldsymbol{D}(K)$, $u_n\to u$, $Ku_n\to v$. We may write

$$(Kw)(x) = F_x[w] \quad \text{for almost all } x,\ w\in\boldsymbol{D}(K),$$

where F_x is a family of bounded linear functionals in $\boldsymbol{L}_2(a, b)$. Obviously

$$(Ku_n)(x) - F_x[u] = F_x[u_n] - F_x[u] \to 0 \quad \text{for almost all } x.$$

By assumption $Ku_n \to v$ in $L_2(a, b)$; hence there exists a subsequence (u_{n_k}) of (u_n) such that $(Ku_{n_k})(x) - v(x) \to 0$ for almost all x. This implies that $v(x) = F_x[u]$ for almost all x, i.e. $u \in D(K)$ and $Ku = v$.

THEOREM 3. *Every strong Carleman operator is a Hilbert-Schmidt operator.*

PROOF. Let K be a strong Carleman operator; then K is closed by Theorem 2. Hence by Theorem III $K = UT$ where T is selfadjoint and U is a partial isometry with initial set $\mathrm{Cl}(R(T))$ and final set $\mathrm{Cl}(R(K))$. Then U^*U is a partial isometry with initial and final set $\mathrm{Cl}(R(T))$, hence $T = U^*K$. By Theorem I T is a selfadjoint strong Carleman operator and consequently by Theorem 1 T is of Hilbert-Schmidt type. Hence $K = UT$ is a Hilbert-Schmidt operator.

3. Remarks on operators which are not densely defined. It is possible to neglect "densely defined" in the definition of a Carleman operator. The kernel $K(x, y) = g(x)h(y)$ $(g \notin L_2(a, b),\ h \in L_2(a, b))$ for example defines a Carleman operator of this type:

$$\begin{aligned} Kf &= 0 \quad \text{if } (f, h) = 0 \\ &= \text{not defined} \quad \text{if } (f, h) \neq 0. \end{aligned}$$

The main disadvantage of these operators is the fact that the kernel is not uniquely determined by the operator (in the above example, g is an arbitrary function not contained in $L_2(a, b)$).

Let K be a strong Carleman operator (in the corresponding sense, i.e. not necessarily densely defined) then KP is a strong Carleman operator, where P is the orthogonal projection onto $\mathrm{Cl}(D(K))$. Since KP is densely defined we may apply the results of §2 and find: *KP is a Hilbert-Schmidt operator.*

REFERENCES

1. T. Carleman, *Sur les équations intégrales singulières à noyau réel et symétrique*, Lundequistska Bokhandeln, Uppsala, 1923.

2. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.

3. J. von Neumann, *Charakterisierung des Spektrums eines Integraloperators*, Hermann, Paris, 1935.

4. M. Schreiber, *Semi-Carleman operators*, Acta Sci. Math. **24** (1963), 82–86.

5. M. Stone, *Linear transformations in Hilbert Space*, Amer. Math. Soc. Colloq. Publ., vol. 15, Amer. Math. Soc., Providence, R. I., 1932.

6. G. Targonski, *Seminar on functional operators and equations*, Springer-Verlag, Berlin, 1967.

INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITY OF HEIDELBERG