ADDITIVE CATEGORIES AND A THEOREM OF W. G. LEAVITT[1]

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The purpose of this note is to present a new proof of the following theorem of Leavitt.

THEOREM 1. *Let K be a commutative ring (with 1), and q a positive integer. Then there exists a K-algebra R such that, for positive integers m, n,*

$$R^m \cong R^n \quad \text{if and only if} \quad m \equiv n \pmod{q}.$$

(R^m denotes the free right R-module on m generators.)

We remark at the outset that as in any case K admits a homomorphism onto a field, an obvious change-of-ring argument shows that it is enough to prove that the theorem is valid in the case where K is itself a field.

Three proofs have already been published, by Leavitt [5], by myself [2], and by Cohn [1]. All three take R to be the K-algebra which is in a fairly obvious sense universal for the isomorphism $R^{q+1} \cong R^1$. To complete the proof, Leavitt shows that $R^m \cong R^n$ only if $m \equiv n$ (mod q) by means of a long and involved cancellation argument. As he is interested in rings rather than algebras he presents his proof for the case where K is the field of 2 elements; but it should be remarked that with a little patience his argument can be adapted to the case where K is an arbitrary field. The remaining two proofs, which are considerably shorter and simpler, avoid the cancellation argument with the help of suitable 'trace functions,' but they have the disadvantage that they fail completely unless the identity element of K/qK has additive order exactly q.

The present proof has none of the disadvantages of the earlier proofs. Tackling the problem from an entirely new angle—it makes no use of the universal algebra already mentioned—it employs a quite trivial manipulation of infinite matrices to deduce the theorem from the familiar invariance of dimension of finite-dimensional vector spaces: the underlying idea is strikingly similar to that exploited by Hanf in his solution of an analogous problem for Boolean algebras [4].

Although it would be possible to present the proof in the manner of a nineteenth-century text on Matrix Theory, it is notationally more convenient to obtain the theorem as a consequence of an embedding theorem for additive categories. For this we need a few *ad hoc* definitions. Let K be a commutative ring, and let $\mathfrak{C}$ be a K-*category*, by which we mean an additive category such that, for objects A, B, C, the groups of morphisms $\mathfrak{C}[A, B]$, $\cdots$ are K-modules and multiplication $\mathfrak{C}[A, B] \times \mathfrak{C}[B, C] \to \mathfrak{C}[A, C]$ is bilinear; note that $\mathfrak{C}[A, A]$ is then always a K-algebra. An object $A(\neq 0)$ of $\mathfrak{C}$ will be called *principal* if the minimal sub-K-algebra $K \cdot 1_A$ of $\mathfrak{C}[A, A]$ is a principal-ideal domain (or field); a nonzero morphism of $\mathfrak{C}$ will be called *principal* if it belongs to $K \cdot 1_A$ for some principal object A.

THEOREM 2. *Let K be a commutative ring, and let $\mathfrak{C}$ be a K-category with finite direct sums. Then $\mathfrak{C}$ may be embedded as a full subcategory of a K-category $\mathfrak{C}^*$ with finite direct sums which contains, for each principal object U of $\mathfrak{C}$, an object U^* such that*
   (i) $U^* \cong U \oplus U^* \cong U^* \oplus U^*$,
   (ii) *for objects A, B of $\mathfrak{C}$,*

$$A \oplus U^* \cong B \oplus U^* \qquad (in\ \mathfrak{C}^*)$$

if and only if there exist nonnegative integers r, s such that

$$A \oplus U^r \cong B \oplus U^s \qquad (in\ \mathfrak{C}).$$

To see that Theorem 1 follows from Theorem 2, let K be an arbitrary field or, more generally, a principal-ideal domain, and take $\mathfrak{C}$ to be the K-category of all free K-modules of finite rank. Given a positive integer q, let $U = K^q$, $A = K^1 \oplus U^*$, $R = \mathfrak{C}^*[A, A]$; note that U is indeed principal, because $K \cdot 1_U = K$. Now, in view of the standard matrix representations for homomorphisms between free right R-modules and for morphisms between direct sums of copies of the object A, it is clear that, for positive integers m, n, the following three conditions are equivalent:
   (a) $R^m \cong R^n$;
   (b) there exist an $m \times n$ matrix X and an $n \times m$ matrix Y over R such that $XY = 1$, $YX = 1$;
   (c) $A^m \cong A^n$.
But it follows at once from (i) that $A^m \cong K^m \oplus U^*$, $A^n \cong K^n \oplus U^*$. Therefore (ii) implies that (c) is equivalent to the existence of nonnegative integers r, s such that $K^m \oplus U^r \cong K^n \oplus U^s$ or, equivalently, taking ranks, such that $m + rq = n + sq$. Visibly, a necessary and sufficient condition for this is that $m \equiv n \pmod q$, as required for the proof of Theorem 1.

PROOF OF THEOREM 2. For the objects of $\mathfrak{C}^*$ we take all families $A = (A_i)_{i \in I}$ of objects of $\mathfrak{C}$ with the property that A_i is principal for almost all $i \in I$. For objects $A = (A_i)_{i \in I}$, $B = (B_j)_{j \in J}$ of $\mathfrak{C}^*$, a morphism $A \rightarrow B$ is to be a row-and-column-finite matrix $X = (x_{ij})_{i \in I, j \in J}$ with $x_{ij} \in \mathfrak{C}[A_i, B_j]$, subject to the requirement that all but a finite number of the nonzero entries x_{ij} be principal. A straightforward verification shows that $\mathfrak{C}^*$ is a K-category under the usual rules for operating with matrices, the identity morphism 1_A of A being, of course, the matrix with the identity morphisms of the A_i in the appropriate positions along the diagonal and zeros elsewhere. Clearly $\mathfrak{C}$ may be identified with the full subcategory of $\mathfrak{C}^*$ whose objects are indexed by the one-element set $\{0\}$. It is equally clear that two objects of $\mathfrak{C}^*$ which differ only in their indexing sets are isomorphic. Therefore, in order to check that two objects $A = (A_i)_{i \in I}$, $B = (B_j)_{j \in J}$ of $\mathfrak{C}^*$ have a direct sum in $\mathfrak{C}^*$, we may assume that $I \cap J = \emptyset$. Take $C = (C_k)_{k \in I \cup J}$, where $C_i = A_i (i \in I)$, $C_j = B_j (j \in J)$, and let X and Y [P and Q] be the matrices obtained from 1_C by retaining only the rows [columns] indexed by the elements of I and J, respectively. Then C is an object and X, Y, P, Q are morphisms of $\mathfrak{C}^*$; and since

$$\begin{bmatrix} X \\ Y \end{bmatrix} [P \quad Q] = \begin{bmatrix} 1_A & \\ & 1_B \end{bmatrix}, \qquad [P \quad Q] \begin{bmatrix} X \\ Y \end{bmatrix} = 1_C$$

we have proved that C is the direct sum of A and B in $\mathfrak{C}^*$ (see Mitchell [6, Proposition I.18.1]).

Given a principal object U of $\mathfrak{C}$, take $U^* = (U_1, U_2, \cdots)$, where $U_i = U$ ($i = 1, 2, \cdots$). It follows at once from what we have just proved that U^* satisfies (i) and the 'if' part of (ii) in the statement of Theorem 2. For the 'only if' part of (ii), let A, B be objects of $\mathfrak{C}$ such that

$$\overline{A} = (A_0, A_1, \cdots) = A \oplus U^* \cong B \oplus U^* = (B_0, B_1, \cdots) = \overline{B},$$

where $A_0 = A$, $B_0 = B$, $A_i = B_i = U$ ($i = 1, 2, \cdots$), and choose inverse isomorphisms $X: \overline{A} \rightarrow \overline{B}$, $Y: \overline{B} \rightarrow \overline{A}$, so that

(1) $XY = 1_{\overline{A}}, \qquad YX = 1_{\overline{B}}.$

By hypothesis, only a finite number of the nonzero entries of X and Y fail to lie in $K \cdot 1_U$; so we may choose a positive integer l such that these exceptional entries all lie in the first l columns of X and the first l rows of Y. Since X and Y are row-and-column-finite, we may further choose positive integers m, n such that the nonzero entries of the first l columns of X [rows of Y] lie in the first m rows of X [columns of Y], and the nonzero entries in the first m rows of X

[columns of Y] lie in the first $l+n$ columns of X [rows of Y]. In this way we obtain block decompositions

$$
(2) \qquad X = \begin{bmatrix} X_0 & X_1 & 0 \\ 0 & X_2 & X_3 \end{bmatrix} \begin{matrix} (m) \\ (\infty) \end{matrix}, \qquad Y = \begin{bmatrix} Y_0 & 0 \\ Y_1 & Y_2 \\ 0 & Y_3 \end{bmatrix} \begin{matrix} (l) \\ (n) \\ (\infty) \end{matrix},
$$
$$
\quad\;\; (l)\;\;(n)\;\;(\infty) \qquad\qquad\qquad\quad (m)\;\;(\infty)
$$

where the bordering parentheses indicate the numbers of rows and columns in the blocks. From (1) and (2) we may read off the equations

$$
Y_1 X_1 + Y_2 X_2 = 1,
$$
$$
X_2 Y_1 = 0, \qquad X_1 Y_2 = 0,
$$

from which it follows easily that

$$
(3) \qquad X_1 \cdot Y_1 X_1 = X_1, \qquad Y_1 X_1 \cdot Y_1 = Y_1,
$$

$$
(4) \qquad X_2 \cdot Y_1 X_1 = 0, \qquad Y_1 X_1 \cdot Y_2 = 0.
$$

By (3), the $n \times n$ matrix $Y_1 X_1$ is clearly idempotent, and our choice of l guarantees that its entries all lie in the principal-ideal domain $K \cdot 1_U$; so there exists an invertible $n \times n$ matrix S over $K \cdot 1_U$ such that, for some nonnegative integer $r \leq n$,

$$
(5) \qquad S^{-1} Y_1 \cdot X_1 S = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \begin{matrix} (r) \\ (n-r) \end{matrix}.
$$

Introducing the automorphism

$$
T = \begin{bmatrix} 1 & & \\ & S & \\ & & 1 \end{bmatrix} \begin{matrix} (l) \\ (n) \\ (\infty) \end{matrix}
$$

of $\bar{B}$, we now replace X, Y by the inverse isomorphisms XT, $T^{-1}Y$. The effect of this is to replace X_1, X_2, Y_1, Y_2 by $X_1 S$, $X_2 S$, $S^{-1} Y_1$, $S^{-1} Y_2$, respectively, and to leave the remaining blocks in the decompositions (2) unchanged. With the new X and Y, (5) takes on the simpler form

$$
(6) \qquad Y_1 X_1 = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \begin{matrix} (r) \\ (n-r) \end{matrix}.
$$

Substitution from (6) in (3) and (4) reveals that the last $n-r$ columns of X_1 [rows of Y_1] and the first r columns of X_2 [rows of Y_2] all vanish, so that by decomposing the central blocks of (2) in the obvious way we obtain finer block decompositions of the form

$$X = \begin{bmatrix} X_0 & X_1' & \\ \hline & X_2' & X_3 \end{bmatrix} \begin{matrix} (m) \\ (\infty) \end{matrix}, \qquad Y = \begin{bmatrix} Y_0 & \\ Y_1' & \\ \hline & Y_2' \\ & Y_3 \end{bmatrix} \begin{matrix} (l+n) \\ \\ (\infty) \end{matrix}.$$
$$\begin{matrix} (l+n) & (\infty) \end{matrix} \qquad\qquad\qquad \begin{matrix} (m) & (\infty) \end{matrix}$$

But then it follows from (1) that the submatrices

$$[X_0\, X_1'] \text{ and } \begin{bmatrix} Y_0 \\ Y_1' \end{bmatrix}$$

are inverse isomorphisms between $(A_0, \cdots, A_{m-1}) = A \oplus U^{m-1}$ and $(B_0, \cdots, B_{l+n-1}) = B \oplus U^{l+n-1}$. The proof of Theorem 2 is complete.

We remark in conclusion that it is not difficult to extend the foregoing argument and show that *a necessary and sufficient condition for two objects* $A = (A_i)_{i \in I}$, $B = (B_j)_{j \in J}$ *of* $\mathfrak{C}^*$ *to be isomorphic is that there exist finite subsets* $I_0 \subseteq I$, $J_0 \subseteq J$, *and a bijection* $f \colon I \backslash I_0 \to J \backslash J_0$ *such that* $\oplus_{i \in I_0} A_i \cong \oplus_{j \in J_0} B_j$ *(in* $\mathfrak{C}$*) and* $A_i = B_{f(i)} (i \in I \backslash I_0)$; we leave the details of the proof to the reader.

For an application of Theorem 1 to primary abelian groups, see [3].

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