A PROOF OF JACKSON'S THEOREM

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1. If f is a continuous function on $[-1, 1]$ and ω_f its modulus of continuity, then the classical theorem of Jackson [1] states that there exists a sequence of polynomials $(J_n[f])$ such that the degree of $J_n[f]$ is $\leqq n$ and

$$\max_{|x|\leqq 1} |J_n[f](x) - f(x)| \leqq C\omega_f\left(\frac{1}{n}\right), \qquad n = 1, 2, \cdots.$$

Various direct, but more or less involved proofs of this result are now available in the literature (see [2], [3], [4], [5], [6] and [7]). In [6] it was shown that Legendre polynomials generate approximating polynomials whose deviation from f on $[-1/4, 1/4]$ is of the order $\omega_f(1/n)$, as in Jackson's theorem. In [7] this result was extended to a large class of orthogonal polynomials.

The aim of this paper is to give a short and simple direct proof of Jackson's theorem by combining an inequality for positive linear operators which was proved recently by O. Shisha and B. Mond [8], with the ideas developed in [6] and [7].

Let $T_{2n}(x) = \cos(2n \text{ arc cos } x)$ be the Chebyshev polynomial of degree $2n$, $\alpha_n = \sin(\pi/4n)$ its smallest positive zero and

$$R_n(x) = c_n\left(\frac{T_{2n}(x)}{x^2 - \alpha_n^2}\right)^2,$$

where c_n is chosen so that $\int_{-1}^{1} R_n(t)dt = 1$. Also let

$$\|g\| = \sup\{|g(x)|: |x| \leqq 1/4\}.$$

We shall prove here the following theorem.

If f is a continuous function on $[-1/2, 1/2]$, then the polynomial $K_n[f]$ defined by

$$K_n[f](x) = \int_{-1/2}^{1/2} f(t)R_n(t - x)dt \tag{1}$$

satisfies the inequality

$$\|K_n[f] - f\| \leqq 2\omega_f\left(\frac{1}{n}\right) + 16\|f\|\frac{1}{n^2}, \qquad n = 1, 2, \cdots. \tag{2}$$

In order to obtain from (2) a proof of Jackson's theorem for the interval $[-1/4, 1/4]$, it is sufficient to consider the modified polynomials $\overline{K}_n[f]$ defined by $\overline{K}_n[f]=f(0)+K_n[f-f(0)]$. Using (2) and elementary properties of the modulus of continuity, we find that for $n\geqq 3$

$$\|\overline{K}_n[f]-f\| \leqq 2\omega_f\left(\frac{1}{n}\right)+16\omega_f\left(\frac{1}{4}\right)\frac{1}{n^2} \leqq 4\omega_f\left(\frac{1}{n}\right).$$

2. In order to simplify the proof of the theorem, we shall first prove the following result.

LEMMA. *For $n=1, 2, \cdots$ we have $\int_{-1}^{1}t^2R_n(t)dt\leqq 1/n^2$.*

PROOF OF THE LEMMA. We have first

$$\int_{-1}^{1} t^2R_n(t)dt \leqq \int_{-1}^{1}(1-t^2)^{-1/2}t^2R_n(t)dt. \tag{3}$$

Next, by Gauss quadrature formula based on the zeros of T_{2n}, we have for any polynomial P of degree $\leqq 4n-1$

$$\int_{-1}^{1}(1-t^2)^{-1/2}P(t)dt = \frac{\pi}{2n}\sum_{k=1}^{2n}P\left(\cos\frac{2k-1}{4n}\pi\right)$$

(see [9, p. 115]). Since R_n is an even polynomial of degree $4n-4$, vanishing at all zeros of T_{2n} except at α_n and $-\alpha_n$, it follows that

$$\begin{aligned}\int_{-1}^{1}(1-t^2)^{-1/2}t^2R_n(t)dt &= \frac{\pi}{n}\alpha_n^2R_n(\alpha_n)\\ &= \alpha_n^2\int_{-1}^{1}(1-t^2)^{-1/2}R_n(t)dt\\ &= \alpha_n^2\int_{-1}^{1}(1-t^2)^{1/2}R_n(t)dt\\ &\quad + \alpha_n^2\int_{-1}^{1}(1-t^2)^{-1/2}t^2R_n(t)dt,\end{aligned}$$

i.e.,

$$(1-\alpha_n^2)\int_{-1}^{1}(1-t^2)^{-1/2}t^2R_n(t)dt = \alpha_n^2\int_{-1}^{1}(1-t^2)^{1/2}R_n(t)dt.$$

Since $\int_{-1}^{1}R_n(t)dt=1$ and $\alpha_n=\sin(\pi/4n)$, it follows that

$$\int_{-1}^{1} (1 - t^2)^{-1/2} t^2 R_n(t)dt \leqq \tan^2 (\pi/4n) \leqq 1/n^2$$

and the lemma is proved in view of the inequality (3).

PROOF OF THE THEOREM. The operator K_n defined by (1) is clearly a positive linear operator. The inequality of Shisha and Mond mentioned earlier states that

$$\|K_n[f] - f\| \leqq (1 + \|K_n[1]\|)\omega_f(\mu_n) + \|f\| \cdot \|K_n[1] - 1\| \tag{4}$$

where $\mu_n = \|K_n[(t-x)^2](x)\|^{1/2}$. Here, the operator K_n is applied to the variable $t \in [-1/2, 1/2]$, while the sup norm $\| \ \|$ is taken with respect to the variable $x \in [-1/4, 1/4]$ (see [8, Theorem 1]). Hence, we have only to evaluate μ_n, $\|K_n[1] - 1\|$ and $\|K_n[1]\|$.

We have, first, for $|x| \leqq 1/4$

$$K_n[(t - x)^2](x) = \int_{-1/2}^{1/2} (t - x)^2 R_n(t - x)dt \leqq \int_{-1}^{1} t^2 R_n(t)dt$$

and so by the lemma

$$\mu_n^2 \leqq \int_{-1}^{1} t^2 R_n(t)dt \leqq 1/n^2. \tag{5}$$

Next,

$$\begin{aligned} 1 - K_n[1](x) &= \int_{-1}^{1} R_n(t)dt - \int_{-1/2}^{1/2} R_n(t - x)dt \\ &= \int_{-x+1/2}^{1} R_n(t)dt + \int_{-1}^{-x-1/2} R_n(t)dt. \end{aligned}$$

Hence, for $|x| \leqq 1/4$ we have

$$\begin{aligned} |1 - K_n[1](x)| &\leqq \left(\int_{1/4}^{1} + \int_{-1}^{-1/4}\right) R_n(t)dt \\ &\leqq 16\left(\int_{1/4}^{1} + \int_{-1}^{-1/4}\right) t^2 R_n(t)dt \end{aligned}$$

and so again by the lemma

$$\|1 - K_n[1]\| \leqq 16 \int_{-1}^{1} t^2 R_n(t)dt \leqq 16/n^2. \tag{6}$$

Finally, for $|x| \leqq 1/4$

$$K_n[1](x) \leqq \int_{-1}^{1} R_n(t)dt = 1 \tag{7}$$

and (2) follows from (4), (5), (6) and (7).

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