CROSS SECTIONALLY CONTINUOUS SPHERES IN E^3

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W. T. Eaton [5] and Norman Hosay [6] independently solved a problem raised by Alexander [1] when they proved that cross sectionally simple spheres in E^3 are tame. A 2-sphere S in E^3 is *cross secticnally simple* if the intersection of S with each horizontal plane $P_t = \{(x, y, z) | z = t\}$ is either empty, a point, or a simple closed curve. The purpose of this note is to show that Eaton's proof, adjusted slightly to incorporate recent results by J. W. Cannon [4], actually shows that cross sectionally continuous spheres are tame. A 2-sphere S in E^3 is *cross sectionally continuous* if each $S \cap P_t$ is a locally connected continuum. The question about the tameness of cross sectionally connected spheres remains open [2].

As in [5] we assume that the cross sectionally continuous sphere S intersects P_t if and only if $-1 \leqq t \leqq 1$, and we denote the continuum $P_t \cap S$ by J_t.

There are two observations to make before we proceed to details of the adjustment of Eaton's proof. First we observe that at most countably many J_t fail to be simple closed curves. This is because a locally connected continuum that is not a simple closed curve must contain a simple triod, and S cannot contain uncountably many disjoint triods [7]. The second observation is that a tame nondegenerate continuum J_t on S is a taming set; that is, each 2-sphere containing J_t and locally tame modulo J_t is tame [4]. In the following paragraphs, we indicate briefly how to incorporate these two observations into Eaton's proof to establish

THEOREM 1. *Cross sectionally continuous 2-spheres in E^3 are tame.*

Let R be a countable subset of $[-1, 1]$ such that

(1) if J_t is not a simple closed curve, then $t \in R$ and

(2) R contains a subset X dense in $[-1, 1]$ such that for $t \in X$, J_t is a simple closed curve, and let $Q = [-1, 1] - R$.

One may think of R as the set of rational numbers in $[-1, 1]$. Lemma 1 of [5] remains valid for cross sectionally continuous spheres as long as t is restricted to Q, and Lemma 2 of [5] is retained as it stands.

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The three steps given in the proof of Lemma 3 of [5] require little change here. Eaton's Step I is used here for each degenerate J_i ($i=1, -1$). Otherwise we skip Step I. To accomplish Step II we use Cannon's result that each J_r ($r \in R$) is a taming set, together with the techniques of [3]. In Step III we again restrict t to Q. No other changes are required.

Actually we have outlined a proof for the following more general theorem.

THEOREM 2. *If each horizontal cross section of a 2-sphere S in E^3 is connected and at most countably many of these cross sections fail to be locally connected, then S is tame.*

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