EUCLIDEAN n-PLANES IN PSEUDO-EUCLIDEAN SPACES AND DIFFERENTIAL GEOMETRY OF CARTAN DOMAINS

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1. **Introduction.** The Cartan domains, which we shall define in §3, include among them the four general types of (irreducible) bounded symmetric domains, first studied by E. Cartan [2], [3]. An (essentially unique) invariant Riemannian metric—the Bergman metric—exists on each of these bounded symmetric domains, and the resulting differential geometry has been studied by Siegel [7], Hua [4], [5], Look [6] and others.

In this note we describe how the differential geometry of Cartan domains can be studied neatly and effectively through a study of the Euclidean n-planes in a pseudo-Euclidean $(n+m)$-space of index m. Our results include a geometric interpretation of the Bergman metric, the theorem that domains of the second and third types are totally geodesic submanifolds of a domain of the first type, and ranges of value of the sectional curvature. Only a brief description of the method and results will be given here. The reader will find in this and three other notes [8], [9], [10] the essence of the differential geometry of the eight nonspecial types of irreducible Hermitian symmetric spaces (see [1]).

2. **Euclidean n-planes in a pseudo-Euclidean space.** Let F be the field R of real numbers, the field C of complex numbers, or the field H of real quaternions. Let $\{1, i, j, k\}$ be the usual basis of F over R. If $\xi = a_0 + a_1 i + a_2 j + a_3 k$, then

$$\bar{\xi} = a_0 - a_1 i - a_2 j - a_3 k, \qquad \xi^\tau = a_0 + a_1 i + a_2 j - a_3 k$$

are two conjugates of ξ. If A is an $n \times m$ matrix with elements in F, we denote by A^*, A^τ the two respective conjugate transposes of A. For a square matrix A, if $A^* = A$, $A^\tau = A$, or $A^\tau = -A$, we say, respectively, that A is Hermitian, τ-symmetric, or τ-skew-symmetric. Clearly, for $F = R$ or C, τ-symmetry and τ-skew-symmetry are the ordinary symmetry and ordinary skew-symmetry.

By definition, a pseudo-Euclidean space $F^{n+m}_{(m)}$ (of index m) is an $(n+m)$-dimensional left vector space over F provided with a (Hermitian) inner product $\langle\ ,\ \rangle$ such that there exist n-planes (i.e. n-dimensional vector subspaces), but not $(n+1)$-planes, on which the induced inner product is positive definite. In $F_{(m)}^{n+m}$, *natural* systems of rectangular coordinates exist such that if

$$(x, y) \equiv (x_1, \cdots, x_n; x_{n+1}, \cdots, x_{n+m})$$

are the coordinates of a vector u, then $\langle u, u\rangle = xx^* - yy^*$.

An important case of $F_{(m)}^{n+m}$ is the real hyperbolic plane $R_{(1)}^2$. If u, v are two vectors of $R_{(1)}^2$ such that $\langle u, u\rangle$, $\langle u, v\rangle$ and v, $\langle v\rangle$ are all >0, then there exists a unique real number θ, called the *angle* between u and v, defined by

$$\cosh \theta = \langle u, v\rangle / (\langle u, u\rangle\langle v, v\rangle)^{1/2}, \qquad 0 \leqq \theta < +\infty.$$

In $F_{(m)}^{n+m}$, an n-plane is called a *Euclidean n*-plane if the inner product induced on it is positive definite. Let A an dB be two Euclidean n-planes in $F_{(m)}^{n+m}$. We can prove that, if u is a nonzero vector in A, and v the orthogonal projection of u in B, then

(i) $v \neq 0$; and

(ii) either $v = u$, or u and v span an $R_{(1)}^2$ and $\langle u, v\rangle > 0$. Thus there exists a unique angle between any nonzero vector u in A and its projection in B, and we can define the *angles* between A and B as the stationary values of the angle between u and its projection in B as u runs through A. With this done, the development of the geometry of Euclidean n-planes in the pseudo-Euclidean space $F_{(m)}^{n+m}$ proceeds parallelly to that of the geometry of n-planes in the Euclidean space F^{n+m}. The definitions and results in [8, §2] can be carried over without difficulty. For example, we can prove that there are n angles between two Euclidean n-planes A and B in $F_{(m)}^{n+m}$ and they completely determine the relative position of A and B; moreover, there are orthogonal frames of angle-planes (i.e., real hyperbolic planes containing the angles) associated with A and B, and so on.

3. The Cartan domains. *The first Cartan domain*, denoted by $D_{\mathrm{I}}(F_{(m)}^{n+m})$, is the manifold of Euclidean n-planes in $F_{(m)}^{n+m}$. Let (x, y) be a natural system of rectangular coordinates in $F_{(m)}^{n+m}$. We can prove that an n-plane in $F_{(m)}^{n+m}$ is a Euclidean n-plane iff it has an equation of the form $y = xZ$, where Z is an $n \times m$ matrix such that $I - ZZ^* > 0$ (i.e., the Hermitian matrix $I - ZZ^*$ is positive definite). Thus, $D_{\mathrm{I}}(F_{(m)}^{n+m})$ can be identified with the space of all $n \times m$ matrices Z such that $I - ZZ^* > 0$. The elements of Z serve as coordinates in $D_{\mathrm{I}}(F_{(m)}^{n+m})$.

In $F_{(n)}^{2n}$, the equation

$$xy̅^{\tau} - yx̅^{\tau} = 0, \tag{3.1}$$

where (x, y) and $(\bar{x}, \bar{y})$ are the coordinates of two vectors in $F_{(n)}^{2n}$, determines a null system; and the equation

$$xy^\tau + yx^\tau = 0 \tag{3.2}$$

determines a hyperquadric. *The second Cartan domain*, denoted by $D_{II}(F^{2n}_{(n)})$, is the manifold of all the Euclidean n-planes in $F^{2n}_{(n)}$ each of which is self-polar with respect to the null system (3.1). *The third Cartan domain*, denoted by $D_{III}(F^{2n}_{(n)})$, is the manifold of all the Euclidean n-planes in $F^{2n}_{(n)}$ each lying entirely in the hyperquadric (3.2). It is easy to see that $D_{II}(F^{2n}_{(n)})$ (resp. $D_{III}(F^{2n}_{(n)})$) can be identified with the space of all $n\times n$ τ-symmetric (resp. τ-skew-symmetric) matrices Z such that $I-ZZ^*>0$.

The group of motions in $F^{n+m}_{(m)}$ induces on $D_I(F^{n+m}_{(m)})$ a transitive group $U_I(F^{n+m}_{(m)})$ of motions. The subgroups of $U_I(F^{2n}_{(n)})$ which leave $D_{II}(F^{2n}_{(n)})$ and $D_{III}(F^{2n}_{(n)})$ respectively invariant are also transitive. Thus Cartan domains are homogeneous spaces; in fact, they are symmetric spaces.

We observe that the Cartan domains $D_I(C^{n+m}_{(m)})$, $D_{II}(C^{2n}_{(n)})$, $D_{III}(C^{2n}_{(n)})$ and $D_I(R^{2+n}_{(n)})$ are precisely the four general types of irreducible bounded symmetric domains (see [3], [5, p. 5] and [1, p. 489]).

4. Invariant Riemannian metric and geodesics in Cartan domains.

THEOREM 4.1. *The sum of squares of the n angles between two consecutive Euclidean n-planes in $F^{n+m}_{(m)}$ provides $D_I(F^{n+m}_{(m)})$ with an invariant Riemannian metric whose analytic expression is*

$$ds^2 = \operatorname{Re}\operatorname{Tr}[(I - ZZ^*)^{-1}dZ(I - Z^*Z)^{-1}dZ^*],$$

where Re Tr *denotes the real part of the trace. In particular, for $F=C$, this reduces to the Bergman metric*

$$ds^2 = \operatorname{Tr}[(I - ZZ^*)^{-1}dZ(I - Z^*Z)^{-1}dZ^*].$$

We have thus a nice geometric interpretation of the Bergman metric on bounded symmetric domains of the first type.

THEOREM 4.2. *The differential equation of the geodesics in $D_I(F^{n+m}_{(m)})$ is*

$$\ddot{Z} + 2\dot{Z}Z^*(I - ZZ^*)^{-1}\dot{Z} = 0,$$

where the dots denote derivatives with respect to the arc length s.

THEOREM 4.3. *Any geodesic in $D_I(F^{n+m}_{(m)})$, $D_{II}(F^{2n}_{(n)})$, or $D_{III}(F^{2n}_{(n)})$ is congruent respectively to*

(i) $Z = \begin{bmatrix} Z_1(s) & 0 \\ 0 & 0 \end{bmatrix}$, $\quad Z_1(s) = \operatorname{diag}(\tanh \tau_1 s, \cdots, \tanh \tau_r s)$,

(ii) $Z = Z(s) = \operatorname{diag}(\pm \tanh \tau_1 s, \cdots, \pm \tanh \tau_r s, 0, \cdots, 0)$, or

$$\text{(iii)}\quad Z = Z(s) = \operatorname{diag}\left\{\tanh \tau_1 s \begin{bmatrix} \cos\omega_1 k & \sin\omega_1 \\ -\sin\omega_1 & -\cos\omega_1 k \end{bmatrix}, \cdots, \right.$$

$$\tanh \tau_q s \begin{bmatrix} \cos\omega_q k & \sin\omega_q \\ -\sin\omega_q & -\cos\omega_q k \end{bmatrix},$$

$$\left. \pm (\tanh \tau_{2q+1} s)k, \cdots, \pm (\tanh \tau_r s)k, 0, \cdots, 0 \right\},$$

where in (i) *and* (ii) *the* τ's *are positive numbers such that* $(\tau_1)^2 + \cdots + (\tau_r)^2 = 1$, *and in* (iii) *the* τ's *and* ω's *are positive numbers such that* $2(\tau_1)^2 + \cdots + 2(\tau_q)^2 + (\tau_{2q+1})^2 + \cdots + (\tau_r)^2 = 1$ *and each of the* ω's *is* $< \pi$.

THEOREM 4.4. *A* C^2*-curve* Γ *in* $D_{\mathrm{I}}(F^{n+m}_{(m)})$ *is a geodesic iff when it is viewed as* 1*-parameter family of Euclidean n-planes in* $F^{n+m}_{(m)}$,

(a) *all the pairs of Euclidean n-planes of* Γ *have common angle-planes, and*

(b) *the n angles* (*arranged in a definite order*) *between any two Euclidean n-planes of* Γ *are proportional to a fixed set of* (*nonnegative*) *constants.*

THEOREM 4.5. (a) *There is a unique geodesic segment joining any two points in* $D_{\mathrm{I}}(F^{n+m}_{(m)})$ (for $F = C$, this is known; see [6]).

(b) *The geodesic segment joining the two points* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *in* $D_{\mathrm{I}}(F^{n+m}_{(m)})$ *is of length* $[\Sigma(\theta_i)^2]^{1/2}$, *where* θ_i *are the n angles between the Euclidean n-planes* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *in* $F^{n+m}_{(m)}$.

The geodesics in $D_{\mathrm{II}}(F^{2n}_{(n)})$ and $D_{\mathrm{III}}(F^{2n}_{(n)})$ also have the properties stated in Theorems 4.4 and 4.5. However, the following inclusive theorem can be proved.

THEOREM 4.6. $D_{\mathrm{II}}(F^{2n}_{(n)})$ *and* $D_{\mathrm{III}}(F^{2n}_{(n)})$ *are totally geodesic submanifolds of* $D_{\mathrm{I}}(F^{2n}_{(n)})$.

Two Euclidean n-planes in $F^{2n}_{(n)}$ are said to be *mutually isoclinic* if the angles between them are all equal. We can prove

THEOREM 4.7. *Any maximal set of mutually isoclinic Euclidean n-planes in* $F^{2n}_{(n)}$ *when viewed as a subset of* $D_{\mathrm{I}}(F^{2n}_{(n)})$ *is a totally geodesic submanifold which is analytically isometric with the pseudo-sphere of curvature* $-4/n$.

5. Sectional curvatures of the Cartan domains. Explicit expression for the sectional curvature of $D_{\mathrm{I}}(F^{n+m}_{(m)})$ differs from that of the Grassmann manifold $G_n(F^{n+m})$ as given in [10, §3] by only a sign. From this expression, we can obtain the ranges of value of the sectional curvature of all the Cartan domains, listed in the following table.

SECTIONAL CURVATURE K

Cartan Domain		Range of Value of K
$D_{\mathrm{I}}(R^{n+m}_{(m)})$	$n=1,\ m=1$	Sectional curvature not defined
	$n=1,\ m\geqq 2$ or $n\geqq 2,\ m=1$	$K=-1$
	$n\geqq 2,\ m\geqq 2$	$-2\leqq K\leqq 0$
$D_{\mathrm{I}}(C^{n+m}_{(m)})$, $D_{\mathrm{I}}(H^{n+m}_{(m)})$	$n=1,\ m=1$	$K=-4$
	$n=1,\ m\geqq 2$ or $n\geqq 2,\ m=1$	$-4\leqq K\leqq -1$
	$n\geqq 2,\ m\geqq 2$	$-4\leqq K\leqq 0$
$D_{\mathrm{II}}(R^{2n}_{(n)})$	$n\geqq 2$	$-2\leqq K\leqq 0$
$D_{\mathrm{II}}(C^{2n}_{(n)})$, $D_{\mathrm{II}}(H^{2n}_{(n)})$	$n\geqq 2$	$-4\leqq K\leqq 0$
$D_{\mathrm{III}}(R^{2n}_{(n)})$	$n=2$	Sectional curvature not defined
	$n=3$	$K=-\frac{1}{2}$
	$n\geqq 4$	$-1\leqq K\leqq 0$
$D_{\mathrm{III}}(C^{2n}_{(n)})$	$n=2$	$K=-2$
	$n=3$	$-2\leqq K\leqq -\frac{1}{2}$
	$n\geqq 4$	$-2\leqq K\leqq 0$
$D_{\mathrm{III}}(H^{2n}_{(n)})$	$n\geqq 2$	$-2\leqq K\leqq 0$

ADDED IN PROOF. The following results can be proved: *The Cartan domains*

$$D_{\mathrm{I}}(R^{n+m}_{(m)}),\quad D_{\mathrm{I}}(C^{n+m}_{(m)}),\quad D_{\mathrm{I}}(H^{n+m}_{(m)});$$
$$D_{\mathrm{II}}(R^{2n}_{(n)}),\quad D_{\mathrm{II}}(C^{2n}_{(n)}),\quad D_{\mathrm{II}}(H^{2n}_{(n)});$$
$$D_{\mathrm{III}}(R^{2n}_{(n)}),\quad D_{\mathrm{III}}(C^{2n}_{(n)}),\quad D_{\mathrm{III}}(H^{2n}_{(n)})$$

have respectively the scalar curvatures

$$-nm(n+m-2),\quad -4nm(n+m),\quad -16nm(n+m+1);$$
$$-\tfrac{1}{2}n(n-1)(n+2),\quad -2n(n+1)^2,\quad -4n(n+1)(2n+1);$$
$$-\tfrac{1}{2}n(n-2)(n-1),\quad -2n(n-1)^2,\quad -4n(n-1)(2n+1).$$

Moreover, with the exception of $D_{II}(R^{2n}_{(n)})$ *and* $D_{III}(H^{2n}_{(n)})$, *they are all Einstein spaces.*

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