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DECOMPOSITIONS OF E^3 INTO POINTS AND COUNTABLY MANY TREES

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In this paper G always denotes a monotone decomposition of E^3, i.e., an upper semicontinuous decomposition into compact connected sets. H_G denotes the set of nondegenerate elements of G, and E^3/G denotes the quotient space of E^3 associated with G. "Homeomorphism" will mean "homeomorphism of E^3 onto itself." If f is a homeomorphism, then $fG=\{f(g)\mid g\in G\}$ and $fG(X)=\bigcup\{f(g)\in fG\mid f(g)$ meets $X\}$. Also, $S(X,r)=\{p\in E^3\mid d(p,X)<r\}$, where d is the usual metric.

The purpose of this paper is to outline some results leading to a proof of the following theorem. Details will be published elsewhere.

THEOREM 1. *If H_G is countable and each element of H_G is a tree consisting of tame arcs, then E^3/G is topologically E^3.*

Recall that a tree is a space homeomorphic to a finite connected one-dimensional simplicial complex containing no simple closed curves. "Consisting of tame arcs" means that each arc of the tree corresponding to a one-simplex is tame. An example given by Fox and Artin [3, p. 987] shows that this condition on a tree is weaker than requiring the tree to be tame.

Theorem 1 extends a result of Bing [2, Theorem 3, p. 370] and answers a question posed by Armentrout [1, p. 5]. Theorem 2 of this paper is the main tool used in the proof of our main result. The methods used to prove Theorem 2 are analogous to those used by McAuley in [4, pp. 444–454].

DEFINITION. Let b be a point of a compact set B in E^3. B is said to be shrinkable to near b with respect to G if given any open set U containing $B\setminus\{b\}$ and any positive number ϵ, there is a homeomorphism h such that

(1) $h=\text{Id}$ on $E^3\backslash U$,

(2) $h(B)\subset S(b,\epsilon)$,

(3) for each $g\in G$ either diam $h(g)<\epsilon$ or $h(g)\subset S(g,\epsilon)$.

The following rather technical lemma essentially says that if B is shrinkable to near b with respect to G and h is a homeomorphism, then there is another homeomorphism f agreeing with h except on a small set and shrinking B to very small size, all without disturbing elements of hG too much. The proof is quite routine.

LEMMA 1. *Let b be a point of a compact set B such that B is shrinkable to near b with respect to G. Let δ and ϵ be positive numbers and h a homeomorphism satisfying*

(1) *for each $g\in G$ either* diam $h(g)<\delta$ *or* $h(g)\subset S(g,\delta)$,

(2) $h(B)\subset U$ *and* $hG(\overline{U})\subset V$, *where U, V are given open sets,*

(3) $h(B\backslash\{b\})\subset 0$, *where 0 is a given open set, and*

(4) $h=\text{Id}$ *on* $E^3\backslash Q$, *where Q is a given bounded open set.*

Then there is a homeomorphism f satisfying

(a) *for each $g\in G$ either* diam $f(g)<\delta+\epsilon$ *or* $f(g)\subset S(g,\delta+\epsilon)$,

(b) $fG(\overline{U})\subset V$,

(c) $f=h$ *on* $E^3\backslash(h^{-1}(0\cap U)\cap Q)$, *and*

(d) $f(B)\subset S(h(b),\epsilon)$.

THEOREM 2. *Let g_0 be an element of G which is the union of compact sets B_0, B_1 and A, where $B_0\cap B_1=\Phi$, $B_i\cap A=\{b_i\}$ $(i=0,1)$, and A is a tame arc joining b_0 and b_1. If B_1 is shrinkable to near b_1 with respect to G, then $B_1\cup A$ is shrinkable to near b_0 with respect to G.*

OUTLINE OF PROOF. Since A is tame it is easy to see that we need only consider the case in which A is a straight line segment. Also, by applying a homeomorphism which pushes things away from A, we may suppose that A is the axis of a solid cylinder T, where $T\cap B_i=\{b_i\}$ $(i=0,1)$ and $T\backslash\{b_0\}\subset V$, where $B_1\cup A\subset V$ open.

We now fit a rectangular cube R_0 closely around A, and then cut R_0 into thin slices with planes $P_0, P_1, \cdots, P_k$, so that we have the situation shown in Figure 1. We use the notation $[P_i,P_j]=\{p\in E^3 \mid p$ is strictly between P_i and $P_j\}$.

Since B_1 is shrinkable to near b_1, there is a homeomorphism f_0 that shrinks B_1 into Int R_0 without disturbing the elements of G very much. Since f_0G is also a monotone decomposition of E^3, there is a rectangular cube R_1 concentric with R_0 such that $A\subset\text{Int } R_1$ and

$$f_0G(R_1)\subset(\text{Int } R_0)\cup(S(B_0,\delta)\backslash T),$$

where δ is very small. Let $\Phi_0=\text{Id}\circ f_0$. Lemma 1 now delivers a homeo-

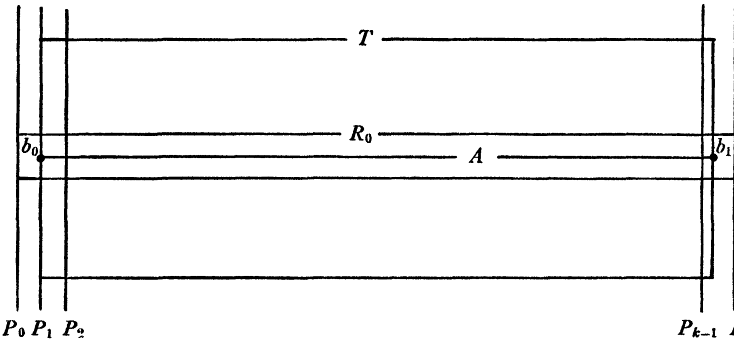

FIGURE 1

morphism f_1 that shrinks B_1 into Int R_1 without disturbing elements of $\Phi_0 G$ very much. Finally, there is a piecewise linear homeomorphism h_1 such that $h_1 = \text{Id}$ on $E^3 \setminus (\text{Int } R_0 \cap [P_{k-2}, P_k])$ and $h_1(R_1) = R_1 \setminus [P_{k-1}, P_k]$. Let $\Phi_1 = h_1 \circ f_1$.

We may now find a rectangular cube R_2 concentric with R_0 such that $\Phi_1(A) \subset \text{Int } R_2$ and

$$\Phi_1 G(R_2) \subset (\text{Int } h_1(R_1) \cap [P_0, P_{k-1}]) \cup (S(B_0, \delta) \setminus T).$$

Just as we found f_1 and h_1 for Φ_0, R_0, and R_1, we now find f_2 and h_2 for Φ_1, $h_1(R_1)$, and R_2. Let $\Phi_2 = h_2 \circ f_2$.

This procedure can be carried on inductively to finally obtain a homeomorphism Φ_{k-2} which shrinks $A \cup B_1$ into $[P_0, P_2] \cap \text{Int } R_0$. Thus if we require the planes P_i to be close together and R_0 to be close to A, we can make diam $\Phi_{k-2}(A \cup B_1)$ as small as we like. Also, if the planes are close enough together and R_0 is close to A, the condition that

$$\Phi_i G(R_{i+1}) \subset (\text{Int } h_i(R_i) \cap [P_0, P_{k-i}]) \cup (S(B_0, \delta) \setminus T)$$

enables us to show that Φ_{k-2} either shrinks g to small size or disturbs g very little for each g in G. Thus Φ_{k-2} is the homeomorphism used to show that $B_1 \cup A$ is shrinkable to near b_0 with respect to G.

The applicability of the following lemma to the case of decompositions into trees should be evident.

LEMMA 2. *Let $B_1, B_2, \cdots, B_n$ be compact sets in E^3 such that $B_i \cap B_j = \{b\}$ for $i \neq j$. Suppose each B_i is shrinkable to near b with respect to G. Then $\bigcup_{i=1}^n B_i$ is shrinkable to near b with respect to G.*

THEOREM 3. *Let g_0 be an element of G which is a tree consisting of tame arcs. Let U be an open set containing g_0, let ϵ be any positive num-*

ber, and let f be any homeomorphism. Then there is a homeomorphism h such that $h=f$ *on* $E^3\backslash U$, diam $h(g_0)<\epsilon$, *and for each g in G either* $h(g)\subset S(f(g), \epsilon)$ *or* diam $h(g)<\epsilon$.

This theorem is easily proved when $f=\mathrm{Id}$ by using an inductive procedure involving Theorem 2 and Lemma 2. It is then a trivial task to prove the general result. Theorem 1 now follows by using the techniques of Bing in [2].

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