ON SMOOTHNESS OF GENTLE PERTURBATIONS[1]

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1. **Introduction.** The notion of gentle perturbations was first introduced by K. O. Friedrichs in connection with some mathematical questions of the quantum theory of fields [1], while the theory of smooth perturbations was introduced by T. Kato in the study of wave operator in scattering theory [2]. It is naturally interesting to find out the relationship between these two concepts. In a recent paper [3, §5] the author was able to prove that every gentle operator of second kind (the class of integral operators with L^1 kernels) relative to some given operator T is a T-doubly smooth operator (see Definition 1.1 below). However, the question whether a T-gentle operator of first kind (the class of operators which can be represented as integral operators with Hölder continuous kernels in a representation in which T is diagonalized) is a T-doubly smooth operator remains unsettled. The purpose of this note is to announce a partial answer to this question. More specifically, we are able to give a simple direct proof that the class of selfadjoint gentle operators of finite rank and the class of nonnegative selfadjoint gentle operators are doubly smooth. Finally we apply our results to obtain some stability theorem on small perturbations.

Throughout this note let H, H' be separable Hilbert spaces and let T be a selfadjoint operator in H with its spectral family $\{E_T(\lambda)\}$. It is known (see [2, Theorem 5.1]) that a closed, densely defined operator A from H to H' with domain $D(A) \supset D(T)$ is T-smooth if $\|A\|_T < \infty$, where

$$\|A\|_T^2 = \sup \|AE_T(I)u\|^2 / |I| \, \|u\|^2,$$

the suprema are taken over all $0 \neq u \in H$ and all semiclosed finite interval $I = (a, b]$ with $|I|$ = the length of I and $E_T(I) = E_T(b) - E_T(a)$.

DEFINITION 1.1. Two closed, densely defined operators A, B from H to H' are said to form a T-doubly smooth pair, if they satisfy the following conditions.

(i) Both A and B are T-smooth, and

(ii) There exists a constant $N < \infty$ such that

$$\|A(T-z)^{-1}B^*u'\| \leq N\|u'\|, \qquad \operatorname{Im} z \neq 0, \; u' \in D(B^*),$$

where B^* denotes the adjoint operator of B.

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An operator K is called a T-doubly smooth operator, if it can be represented formally as the product B^*A with $[A, B]$ a T-doubly smooth pair.

Since the notion of T-smoothness is essentially related to the continuous spectrum of T (cf. [2, Remark 1.8]), in what follows we shall assume, for simplicity, that the spectrum of T is simple and absolutely continuous. Let $S=\operatorname{supp}(T)$ be the support of T (i.e. the smallest Borel subset of the real line for which $E_T(S)=1$; note that S is unbounded for unbounded T). By spectral representation theorem (cf. [4], [5], [1]) T admits a simple direct spectral transformation U (unitary) from H to $L^2(S)$ such that U carries T into the multiplication operator M of $L^2(S)$. That is $T=U^*MU$, where $Mu(x)=xu(x)$ for $u\in D(M)\subset L^2(S)$. The system $\{M, L^2(S)\}$ will be called the spectral representation of the system $\{T, H\}$ under U.

Let $k(x, y)$ be a complex valued function defined on $S\times S$. For $0<\alpha<1$ and $0<\beta<1$, define

$$\begin{aligned}
\Delta_1 k(x, y) &= h^{-\beta}[k(x+h, y) - k(x, y)],\\
\Delta_2 k(x, y) &= h^{-\beta}[k(x, y+h) - k(x, y)],\\
Q_1 k(x, y) &= (1+|x|^\alpha)k(x, y),\\
Q_2 k(x, y) &= (1+|y|^\alpha)k(x, y)
\end{aligned}$$

and set

$$\|k\|_{\alpha,\beta} = \sup \max\{|Q_1Q_2k(x, y)|, |Q_1\Delta_2k(x, y)|, |Q_2\Delta_1k(x, y)|, |\Delta_1\Delta_2k(x, y)|\}$$

where the suprema are taken over $(x, y)\in S\times S$ and $0<h$.

The collection of all such functions k with $\|k\|_{\alpha,\beta}<\infty$ is denoted by $R(\alpha, \beta, S)$.

DEFINITION 1.2. Let $\{M, L^2(S)\}$ be the spectral representation of $\{T, H\}$ under U. An integral operator K from $L^2(S)$ to $L^2(S)$ with kernel $k(x, y)$ is said to be M-gentle, if $\|k\|_{\alpha,\beta}<\infty$ for some $0<\alpha<1$ and $0<\beta<1$. The class of such operators is also denoted by $R(\alpha, \beta, S)$.

This is an extension of the first gentle space of Friedrichs [4], [5]. The properties of such classes are given in [6], [7].

DEFINITION 1.3. An operator K from $L^2(S)$ to $L^2(S)$ is said to be M-absolutely gentle, if its absolute value $|K|=(K^*K)^{1/2}$ is M-gentle.

REMARK 1.4. (1) Evidently every nonnegative selfadjoint M-gentle operator K is M-absolutely gentle. For in this case $K=|K|$.

(2) Let $K\in R(\alpha, \beta, S)$ be selfadjoint of finite rank. Then K can be written in its dyadic form $K=\sum e_j(\ ,f_j)f_j$ $(j=1, 2, \cdots n)$, where the

$\{f_j\}$ are the eigenfunctions of K and the $\{e_j\}$ are the corresponding eigenvalues. It is known (cf. [6, Appendix III]) that $k(x,y) = \sum e_j f_j(x)[f_j(y)]^-$ belongs to $R(\alpha, \beta, S)$ implies that $f_j \in R(\alpha, \beta, S)$ (as functions of one variable). The kernel $\mathbf{k}$ of $|K|$ is precisely $\mathbf{k}(x, y) = \sum |e_j| f_j(x) [f_j(y)]^-$. Since $R(\alpha, \beta, S)$ is a linear space (complete in its norm), it follows that $\mathbf{k} \in R(\alpha, \beta, S)$. Thus K is M-absolutely gentle.

2. **The main results.** Let $\{M, L^2(S)\}$ be the spectral representation of $\{T, H\}$ under U throughout this section.

THEOREM 2.1. *If K is a selfadjoint M-absolutely gentle operator in $L^2(S)$, then the corresponding operator UKU^* is a T-doubly smooth operator in H.*

THEOREM 2.2. *Let $K \in R(\alpha, \beta, S)$, $0<\alpha<1$, $0<\beta<1$, be a selfadjoint M-gentle operator of finite rank. Then there exists a constant $N<\infty$ depending on α, β and $\|k\|_{\alpha,\beta}$ such that for any complex number ϵ with $|\epsilon| <1/N$, the operator $M+\epsilon K$ is spectrally absolutely continuous and is unitarily equivalent to M.*

THEOREM 2.3. *The result of Theorem 2.2 holds true if one replaces "selfadjoint M-gentle operator of finite rank" by "nonnegative selfadjoint M-gentle operator".*

Theorem 2.2 and 2.3 are stability theorems on small gentle perturbations. Their proofs follow easily from Remark 1.4, Theorem 2.1 and a result on smooth perturbations due to Kato [2, Theorem 1.5].

The proof of Theorem 2.1 is based on the following lemmas. We shall give an outline of their proofs. Full details will appear elsewhere.

LEMMA 2.4. *A pair of operators $[A, B]$ is T-doubly smooth in H if and only if its representation $[U^*AU, U^*BU]$ is M-doubly smooth in $L^2(S)$.*

This lemma is an immediate consequence of the following identities.

$$\begin{aligned} \|U^*AUE_M(I)\|^2 &= \|AUE_M(I)\|^2 = \|A[UE_M(I)U^*]A^*\| \\ &= \|AE_T(I)A^*\| = \|AE_T(I)\|^2, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \|U^*AU(M-z)^{-1}U^*BU\| &= \|A[U(M-z)^{-1}U^*]B\| \\ &= \|A[U^*(M-z)U]^{-1}B\| \\ &= \|A(T-z)^{-1}B\|, \qquad \operatorname{Im} z \neq 0. \end{aligned} \tag{2}$$

LEMMA 2.5. *Every selfadjoint M-absolutely gentle operator K can be written as the product BA of two selfadjoint operators A, B with the property that the pair $[A, B]$ is M-doubly smooth in $L^2(S)$.*

PROOF. Let $K=W|K|$ be the polar decomposition of K. Since K is selfadjoint, we have $W=W^*$ and $W^2=1$, moreover W commutes with K and $|K|$ (cf. [8, pp. 334–336]). Set $A=|K|^{1/2}$ and $B=W|K|^{1/2}$; thus $K=BA$, A and B both selfadjoint. By our assumption $|K|$ is an integral operator with kernel $\boldsymbol{k}\in R(\alpha, \beta, S)$ for some $0<\alpha<1$ and $0<\beta<1$. Now we use the notion of Schmidt norm to estimate the operator $AE_M(I)$, for each $I\subset S$. We found that $\|AE_M(I)\|_2^2\leqq C|I|$, where $\|\cdot\|_2$ denotes the Schmidt norm and $C=\sup|\boldsymbol{k}(x, y)|$ over $(x, y)\in S\times S$ (which is attained since $\boldsymbol{k}$ is gentle). It follows that $\|A\|_M\leqq C^{1/2}$. Hence A is M-smooth. The M-smoothness of B is proved similarly.

It remains to show the existence of a constant $N<\infty$ for which $\|A(M-z)^{-1}Bu\|\leqq N\|u\|$ holds uniformly in z for $\operatorname{Im} z\neq 0$, $u\in D(B)$. To this end we first show that the derivative of $AE_M(s)B$ with respect to s exists in the uniform operator topology. We denote it by $AE_M'(s)B$. Next we succeeded in proving that the $\mathcal{B}(L^2(S))$ (Banach algebra of all bounded operators on $L^2(S)$) valued function $AE_M'(s)B$ is Hölder continuous with the Hölder exponent β and it vanishes at infinity with the Hölder quotient α in the norm of $\mathcal{B}(L^2(S))$. That is, as a function of s, $AE_M'(s)B$ belongs to the class $R(\alpha, \beta, S)$. We have $\|AE_M'(\cdot)B\|_{\alpha,\beta}\leqq\|\boldsymbol{k}\|_{\alpha,\beta}$. Finally we apply a result of Plemelj-Privalov (see [9, Chapter 2]) to obtain a uniform bound of the operator $A(M-z)^{-1}B$. Recall

THEOREM P (PLEMELJ-PRIVALOV). *Let $g(s)$ be a H-valued function belongs to $R(\alpha, \beta, S)$, $0<\alpha<1$ and $0<\beta<1$. Define*

$$G(z) = \int_S g(s)\cdot(z-s)^{-1}ds, \qquad \operatorname{Im} z \neq 0.$$

Then the function $G(z)$ is Hölder continuous in the norm of H with the same Hölder exponent β. Moreover there exists a constant C depends on α, β such that $\|G(z)\|\leqq C\|g\|_{\alpha,\beta}$.

It is well known that

$$A(M-z)^{-1}Bu = \int_S AE_M'(s)Bu\cdot(z-s)^{-1}ds, \quad \operatorname{Im} z \neq 0, \quad u \in D(B).$$

Since $AE_M'(s)B\in R(\alpha, \beta, S)$, by Theorem P, we therefore have

$$\|A(M-z)^{-1}Bu\| \leqq C\|AE'_M(\cdot)Bu\|_{\alpha,\beta} \leqq C\|AE'_M(\cdot)B\|_{\alpha,\beta}\|u\|,$$

for $u \in D(B)$ and $\operatorname{Im} z \neq 0$. This completes the proof of Lemma 2.5.

Theorem 2.1 follows from Lemma 2.4 and Lemma 2.5.

REMARK 2.6. If the spectrum of T is not simple, then the spectral representation of the system $\{T, H\}$ is a system $\{M, L^2(S, \mathfrak{H})\}$ where $\mathfrak{H}$ is an accessory Hilbert space. Our proofs can be carried over literally to this case (with some obvious modifications).

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