CELLULAR DECOMPOSITIONS OF 3-MANIFOLDS THAT YIELD 3-MANIFOLDS[1]

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1. **Introduction.** The purpose of this note is to announce several results concerning cellular decompositions of 3-manifolds for which the associated decomposition space is a 3-manifold.

In [6], Bing raised the question of whether each point like decomposition of E^3 that yields a 3-manifold yields E^3. This question leads naturally to the following one: Suppose M is a 3-manifold and G is a cellular decomposition of M such that the associated decomposition space is a 3-manifold N. Is N homeomorphic to M? In Theorem 2 below, we give an affirmative answer to this question. Previously, partial solutions appeared in [1], [2], [3], [7], [8], [9].

The main result announced in this note is Theorem 1 below, and the other theorems stated are derived from it. Detailed proofs will appear elsewhere.

Our results have applications to the problem of realizing, in the case considered here, the decomposition space as the final stage of a pseudoisotopy. T. M. Price has established such a result for cellular decompositions of S^3 that yield S^3 [11]. W. Voxman has extended these results to arbitrary 3-manifolds [12].

In another paper [4], we apply the results announced here to the study of cellular decompositions of 3-manifolds with boundary, and derive a result for such manifolds analogous to Theorem 2 below. We also use these results to study shrinkability conditions satisfied by certain cellular decompositions of 3-manifolds that yield 3-manifolds [5]. Our results in this direction have been extended greatly by Voxman [13].

2. **Preliminaries.** The notation and terminology of [3] will be followed here. In particular, if X is a topological space and G is an upper semicontinuous decomposition of X, then X/G denotes the associated decomposition space and P denotes the projection map from X onto X/G.

If M is an n-manifold, the compact set A is *cellular* (in M) if and only if for each neighborhood U of A, there is an n-cell C in M such that $A \subset \operatorname{Int} C$ and $C \subset U$. If M is either E^n or S^n, a compact connected

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set A is *pointlike* (in M) if and only if $M-A$ is homeomorphic to $M-\{0\}$, 0 being some point of M. It is well known that in E^n and S^n, "pointlike" and "cellular" are equivalent.

3. The main result.

THEOREM 1. *Suppose that M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N. Suppose that T is a triangulation of N such that if $\sigma \in T$, there is an open 3-cell U_σ in M such that $P^{-1}[\sigma] \subset U_\sigma$. Then there exist a triangulation Σ of M and an isomorphism ϕ from T onto Σ such that if $\sigma \in T$, $\phi(\sigma) \subset U_\sigma$.*

The proof of Theorem 1 is modeled on that of the main result of [3]. The following lemma is of basic importance to the proof of Theorem 1.

LEMMA. *Suppose M is a triangulated 3-manifold and G is a cellular decomposition of M such that M/G is a triangulated 3-manifold N. Suppose U is an open 3-cell in N such that $P^{-1}[U]$ is an open 3-cell V in M. Suppose that S is a polyhedral 2-sphere in U, τ is a triangulation of S, and for each simplex σ of τ, U_σ is an open set in U containing σ. Then there exist a polyhedral 2-sphere S' in V, a triangulation τ' of S', and an isomorphism ζ from τ onto τ' such that if $\sigma \in \tau$, $\zeta(\sigma) \subset P^{-1}[U_\sigma]$.*

4. Applications.

THEOREM 2. *Suppose M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N. Then M and N are homeomorphic.*

As a corollary of Theorem 2, we have the following result which settles a question raised by Bing [6].

THEOREM 3. *If G is a pointlike decomposition of E^2 such that E^3/G is a 3-manifold, then E^3/G is homeomorphic to E^3.*

THEOREM 4. *Suppose M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N. If U is any open set in N, then $P^{-1}[U]$ is homeomorphic to U.*

The following corollary of Theorem 4 is closely related to Theorem 2.2 of [10].

THEOREM 5. *Suppose M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N. If U is an open 3-cell in N, then $P^{-1}[U]$ is an open 3-cell in M.*

We are now in a position to establish a strengthened version of Theorem 1.

THEOREM 6. *Suppose M is a 3-manifold and G is a cellular decomposition of M such that M/G is a 3-manifold N. Suppose T is a triangulation of N and for each simplex σ of T, U_σ is an open set in M containing $P^{-1}[\sigma]$. Then there is a triangulation Σ of M and an isomorphism ϕ from T onto Σ such that if $\sigma \in T$, $\phi(\sigma) \subset U_\sigma$.*

Under the hypothesis of Theorem 1, we are able to establish a stronger conclusion. The isomorphism ϕ from T onto Σ induces a homeomorphism h from N onto M. We may construct such an h so that, in a certain sense, the composite Ph from N onto N is near the identity, and may be homotopically deformed to the identity by a homotopy with short paths.

THEOREM 7. *Suppose M is a 3-manifold and G is a cellular decomposition of M such that M/G is a triangulated 3-manifold N. Suppose T is a triangulation of N and if $\sigma \in T$, U_σ is an open set in M containing $P^{-1}[\sigma]$. Then there exist*

(1) *a piecewise linear homeomorphism h from N onto M and*

(2) *a homotopy H from $N \times [0, 1]$ into N such that $H_0 = Ph$, H_1 is the identity map from N onto N, and if $\sigma \in T$, $H[\sigma \times [0, 1]] \subset P[U_\sigma]$.*

5. Compact mappings. The results of this note can be formulated in terms of compact mappings. If f is a mapping from a manifold M onto a space Y, then f is *cellular* if and only if for each point y of Y, $f^{-1}[y]$ is cellular.

THEOREM 8. *If f is a compact cellular mapping from a 3-manifold M onto a 3-manifold N, then M and N are homeomorphic.*

REFERENCES

1. S. Armentrout, *Upper semi-continuous decompositions of E^3 with at most countably many non-degenerate elements*, Ann. of Math. (2) 79 (1963), 605–618.

2. ———, *Decompositions of E^3 with a compact 0-dimensional set of nondegenerate elements*, Trans. Amer. Math. Soc. **123** (1966), 165–177.

3. ———, *Concerning cellular decompositions of 3-manifolds that yield 3-manifolds*, Trans. Amer. Math. Soc. **133** (1968), 307–332.

4. ———, *Concerning cellular decompositions of 3-manifolds with boundary*, Trans. Amer. Math. Soc. **137** (1969), 231–336.

5. ———, *Shrinkability of certain decompositions of E^3 that yield E^3*, Illinois J. Math. (to appear).

6. R. H. Bing, *Point-like decompositions of E^3*, Fund. Math. 50 (1962), 431–453.

7. E. H. Connell, *Images of E_n under acyclic maps*, Amer. J. Math. **83** (1961), 787–790

8. R. L. Finney, *Point-like, simplicial mappings of a 3-sphere*, Canad. J. Math. **15** (1963), 591–604.

9. T. M. Price, *Upper semicontinuous decompositions of E^3*, Thesis, University of Wisconsin, Madison, Wis., 1964.

10. ———, *A necessary condition that a cellular upper semicontinuous decomposition of E^n yield E^n*, Trans. Amer. Math. Soc. **122** (1966), 427–435.

11. ———, *Decompositions of* S^3 *and pseudo-isotopies*, Notices Amer. Math. Soc. **15** (1968), 103.

12. W. Voxman, *Decompositions of 3-manifolds and pseudo-isotopies*, Notices Amer. Math. Soc. **15** (1968), 547.

13. ———, *On the shrinkability of decompositions of 3-manifolds*, Notices Amer. Math. Soc. **15** (1968), 649.

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ON ONE PARAMETER FAMILIES OF REAL SOLUTIONS OF NONLINEAR OPERATOR EQUATIONS

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Let H be a separable Hilbert space over the real numbers. Denote by L, M, N bounded mappings of H into itself. In an earlier note [1] the author studied the solutions $u \in H$ of operator equations of the form

$$u + Mu = \lambda\{Lu + Nu\} \tag{1}$$

under the assumption that such solutions be contained in a sufficiently small sphere with center at $u=0$. Here we assume that L, M, and N are odd, completely continuous, locally Lipschitzian gradient mappings of H into itself, and λ is a real parameter. We wish to announce some results concerning the global structure of solutions of (1) and some applications of these results to problems arising in the calculus of variations. Proofs of these results will appear elsewhere.

1. Statements of results. Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are even functionals such that grad $\mathfrak{M}=M$ and grad $\mathfrak{N}=N$, then we set $F(u) =(1/2)\|u\|^2+\mathfrak{M}(u)$ and $G(u)=(1/2)(Lu,u)+\mathfrak{N}(u)$, and $\partial A_R = \{u \mid F(u)=R,\ R \text{ a fixed real number}\}$. Associated with (1) we consider its linearization at $u=0$

$$u = \lambda Lu. \tag{2}$$

We assume $(Lu, u)>0$ for $u\neq 0$ so that the eigenvalues of (2) $\{\lambda_i\}$ form an increasing sequence of real numbers $0<\lambda_1\leqq\lambda_2\leqq\lambda_3 \cdots$.

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