BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

BY ROBERT FENNELL[1] AND PAUL WALTMAN[2]

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Let E^n denote Euclidean n space with norm $|\cdot|$ and let C_h denote the space of continuous E^n valued functions on $[a-h, a]$, $h>0$, with uniform norm $\|\cdot\|$. For a function $x(t)$ on $[a-h, b]$ and $t\in[a, b]$ let x_t denote the function on $[a-h, a]$ whose value at θ is $x(t+\theta-a)$. Let $f(t, \Psi)$ be a mapping from $[a, b]\times C_h$ into E^n and let M and N be linear operators from C_h to C_h. In this paper we consider special cases of the boundary value problem

$$y' = f(t, y_t), \tag{1}$$

$$My_a + Ny_b = 0, \qquad b > a + h. \tag{2}$$

This is a nonlinear version of a problem posed by Cooke [1]. Other boundary value problems for functional differential equations have been studied recently by Grimm and Schmitt [3], Halanay [4], and Kato [6].

We treat the problem for bounded f and a restricted class of operators by initial value methods, that is, we seek to find an initial function $q\in C_h$ such that a solution of the initial value problem (1) and

$$y(t) = q(t), \qquad a - h \leqq t \leqq a \tag{3}$$

satisfies the boundary condition (2). Some functions $f(t, y_t)$, not bounded, can be treated by approximation techniques and some nonhomogeneous boundary conditions can also be considered.

THEOREM 1. *Let $f(t, \Psi)$ be a continuous bounded function from $[a, b]\times C_h$ into E^n and let M and N be $n\times n$ matrices such that $M+N$ is nonsingular. If $\|(M+N)^{-1}N\|<1$, then there exists a solution of* (1) *and* (2).

METHOD OF PROOF. We give a brief sketch of the method of proof of Theorem 1. The proofs of the other theorems are similar. Assume first that solutions of the initial value problem are unique. Let T: $C_h\to C_h$ be defined as follows: for $q\in C_h$, let $Tq=x_b(q)$, that is, Tq is the segment at b of the solution of the initial value problem with

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initial data q. Let

$$S(\alpha, L) = \{q \mid q \in C_h, \|q\| \leqq \alpha, \mid q(\theta_1) - q(\theta_2) \mid \leqq L \mid \theta_1 - \theta_2 \mid, \theta_1, \theta_2 \in [a - h, a]\}.$$

The operator

$$F(q) = [I - (M + N)^{-1}(M + NT)]q = -(M + N)^{-1}N(T - I)q$$

is shown to have a fixed point $q^* \in S(\alpha, L)$ for an appropriate choice of α and L. The solution of the initial value problem with initial data q^*, satisfies the boundary condition (2). An approximation argument, essentially that of Kato [6], is then used to remove the uniqueness assumption.

In case the matrix M is invertible the boundary condition (2) can be written

$$y_a + Py_b = 0. \tag{4}$$

If $\|P\| < 1$, then $I + P$ is nonsingular [2, p. 62] but it is not necessarily the case that $\|(I+P)^{-1}P\| < 1$. The following theorem covers this case.

THEOREM 2. *Let $f(t, \Psi)$ be a continuous bounded function from $[a, b] \times C_h$ into E^n and let P be an $n \times n$ matrix. If $\|P\| < 1$, then the boundary value problem* (1) (4) *has a solution.*

In the preceding theorems, the desired initial function was selected from a compact set $S(\alpha, L)$. We can replace the matrix P in Theorem 2 by a linear operator but we no longer can select the initial condition from a predetermined compact set and thereby lose the approximation argument which allowed the assumption of unique solutions of the initial value problem (1) (3) to be avoided.

THEOREM 3. *Let $f(t, \Psi)$ be a continuous bounded function from $[a, b] \times C_h$ into E_n and let P be a continuous linear operator from C_h into C_h with $\|P\| < 1$. Suppose solutions of the initial value problem are unique. Then there exists a solution of* (1) (4).

THEOREM 4. *Let the hypotheses of Theorem 3 hold with the boundedness assumption on $f(t, \Psi)$ replaced by a Lipschitz condition. If $e^{L(b-a)}(b-a) < (1 - \|P\|)/L\|P\|$, there exists a unique solution of* (1) (4).

Details will appear elsewhere.

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University of Iowa, Iowa City, Iowa 52240