ON BOUNDARY REGULARITY FOR PLATEAU'S PROBLEM[1]

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We state here sufficient conditions for certain minimal surfaces to be differentiable at boundary points.

Let m and n be integers with $1<m<n$. We adopt the notation of [3]. See also [2]. In particular, $I_m(R^n)$ is the group of m dimensional integral currents in R^n. If $T\in I_m(R^n)$, $M(T)$ is the mass of T and ∂T is the boundary of T; if $a\in R^n$, $\Theta^m(\|T\|, a)$ is the m dimensional density of the variation measure $\|T\|$ at a.

If $T\in I_m(R^n)$, we say T is minimal if there exists $r>0$ such that $M(T)\leqq M(S+T)$ whenever $a\in R^n$, $S\in I_m(R^n)$, $\partial S=0$ and spt $S\subset\{x: |x-a|<r\}$. Given $B\in I_{m-1}(R^n)$ with $\partial B=0$, it is shown in [3] that there exists $T\in I_m(R^n)$ such that $\partial T=B$ and $M(T)\leqq M(S+T)$ whenever $S\in I_m(R^n)$ with $\partial S=0$.

THEOREM. *Suppose* $T\in I_m(R^n)$, *T is minimal,* $a\in$spt ∂T, $p\geqq 2$, $\Theta^{m-1}(\|\partial T\|, a)=1$ *and* spt ∂T *intersects some neighborhood of a in a class p (real analytic) $m-1$ dimensional submanifold of* R^n.

(1) *If* $\Theta^m(\|T\|, a)=1/2$, *then the intersection of* spt T *with some neighborhood of a is a subset of some class $p-1$ (real analytic) m dimensional submanifold of* R^n.

(2) *If there exist independent linear functionals* α_i, $i=1,\cdots, n-m+1$, *on R^n such that either*

$$\text{spt}\,\partial T\subset\{x: \alpha_i(x-a)\geqq 0,\ i=1,\cdots,n-m+1\},$$

or there is $r>0$ such that

$$\{x: |x-a|<r\}\cap\text{spt}\,T\subset\{x: \alpha_i(x-a)\geqq 0,\ i=1,\cdots,n-m+1\},$$

then $\Theta^m(\|T\|, a)=1/2$.

COROLLARY. *Suppose $p\geqq 2$ and B is the $m-1$ dimensional integral current corresponding to some compact oriented class p (real analytic) $m-1$ dimensional submanifold N of R^n. If N lies on the boundary of some uniformly convex open subset of R^n and* $T\in I_m(R^n)$ *is minimal*

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with $\partial T = B$ then there is $r > 0$ and a class $p-1$ (real analytic) m dimensional submanifold M of R^n such that

$$\operatorname{spt} T \cap \{x: \text{distance } (x, N) < r\} \subset M.$$

Statement (1) is proved by combining the interior regularity results of [1] or of [8] with the construction of certain surfaces of dimension $n-1$ which are barriers for the m dimensional area problem, and then applying the higher differentiability results of [6] and [7]. Statement (2) is proved by applying a variational argument to a tangent cone of T at a. The corollary is an elementary consequence of the theorem.

These results remain true if we replace the group $I_m(R^n)$ by the group of flat chains over the integers modulo 2, as in [5]. If $m=2$ and $n=3$, the only boundary density that occurs on the smooth boundary of a minimal chain is one half. In view of the interior regularity results of [4], a minimal flat chain over the integers modulo 2 in R^3 which spans a finite family of smooth curves must be free from singularities of any kind.

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